ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

J. Differential Equations 203 (2004) 82–118

Journal of Differential Equations

http://www.elsevier.com/locate/jde

Large time behavior and $L^p - L^q$ estimate of solutions of 2-dimensional nonlinear damped wave equations

Takafumi Hosono[a,*] and Takayoshi Ogawa[b]

[a] Faculty of Mathematics, Graduate School of Mathematics, Kyushu University, Fukuoka 812-8581, Japan

[b] Mathematical Institute, Tohoku University, Sendai, 980–8578, Japan

Received August 4, 2003; revised March 25, 2004

Abstract

We show the asymptotic behavior of the solution to the Cauchy problem of the two-dimensional damped wave equation. It is shown that the solution of the linear damped wave equation asymptotically decompose into a solution of the heat and wave equations and the difference of those solutions satisfies the $L^p - L^q$ type estimate. This is a two-dimensional generalization of the three-dimensional result due to Nishihara (Math. Z. 244 (2003) 631). To show this, we use the Fourier transform and observe that the evolution operators of the damped wave equation can be approximated by the solutions of the heat and wave equations. By using the $L^p - L^q$ estimate, we also discuss the asymptotic behavior of the semilinear problem of the damped wave equation with the power nonlinearity $|u|^\alpha u$. Our result covers the whole super critical case $\alpha > 1$, where the $\alpha = 1$ is well known as the Fujita exponent when $n = 2$.

MSC: 35B33; 35B40; 35L15; 35L70

Keywords: Besov space; Cauchy problem; Critical exponent; Damped wave equation; Fourier transform; Large time asymptotic behavior; $L^p - L^q$ estimate; Power nonlinearity; Self-similar profile; Time-global solvability

*Corresponding author. Fax: +81-92-642-2778.

E-mail addresses: hosono@math.kyushu-u.ac.jp (T. Hosono), ogawa@math.tohoku.ac.jp (T. Ogawa).

0022-0396/$ - see front matter

doi:10.1016/j.jde.2004.03.034

1. Introduction

In this paper, we consider the Cauchy problem for the damped wave equation in two-dimensional space

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = |u|^\alpha u, \quad (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^2, \\ \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $u = u(t,x) : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$, $\alpha > 0$.

The semilinear Cauchy problem (1.1) has been investigated by many authors [10,12–16,18,19,21–24,27,31–33].

It has been conjectured that the damped wave equation has the diffusive structure as $t \to \infty$ (see [1,17]). This suggests that problem (1.1) should have a similar critical exponent as one can observe for the semilinear heat equation. This exponent $\alpha = \frac{2}{n}$ is known as the Fujita exponent (cf. [9]). Indeed, Todorova–Yordanov [31] proved that the critical exponent of (1.1) is exactly the same as the Fujita exponent for the semilinear heat equation $\alpha = \frac{2}{n}$ in general space dimension. This result is obtained by taking the data of a compact support. Ikehata–Miyaoka–Nakatake [13] showed the global existence of the weak solution to (1.1) and its decay when $1 < \alpha < \frac{2}{n-2}$ ($n = 1, 2, 3$) without assuming that the data is compact support. When $n = 3$ and $\alpha > \frac{2}{3}$, Nishihara [25] showed the global existence of the weak solution to (1.1) and its decay. Ono [27] also showed the global existence for $\alpha > \frac{2}{n}$ when $n = 1, 2, 3$ for the data in $(H^1 \cap L^1) \times (L^2 \cap L^1)$ (see also [14]).

To analyse the dissipative structure further more, Nishihara [25] considered the three-dimensional Cauchy problem for the linear damped wave equation and obtained the $L^p - L^q$ estimate of the difference with the solution of corresponding heat and wave equations. More precisely, let $u(t,x)$ be the solution to the following linearized damped wave equation;

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^3, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^3, \\ \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

$v(t,x)$ be the solution of linear heat equation;

$$\begin{cases} \partial_t v - \Delta v = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^3, \\ v(0,x) = u_0(x) + u_1(x), \quad x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

and $w(t,x)$ and $\tilde{w}(t,x)$ be the solution of linear wave equation:

$$\begin{cases} \partial_t^2 w - \Delta w = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^3, \\ w(0,x) = u_0(x), & x \in \mathbb{R}^3, \\ \partial_t w(0,x) = u_1(x), & x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $\tilde{w}(t,x)$ corresponds the solution of the linear wave equation (1.4) with the initial data $w(0,x) = 0$, $\partial_t w(0,x) = u_0(x)$. Then for $t \geqslant 1$,

$$\begin{aligned} &\left\| u(t) - v(t) - e^{-\frac{t}{2}} \left\{ w(t) + \left(\frac{1}{2} + \frac{t}{8} \right) \tilde{w}(t) \right\} \right\|_{L^p} \\ &\quad \leqslant C t^{-\gamma - 1} (\| u_0 \|_{L^q} + \| u_1 \|_{L^q}) \end{aligned} \tag{1.5}$$

with $\gamma = \frac{3}{2}(\frac{1}{q} - \frac{1}{p})$. Marcati–Nishihara [19] also showed the $L^p - L^q$ estimate (1.1) when $n = 1$. This estimate was applied to the semilinear problem (1.1).

Our purpose of this paper is to obtain the similar $L^p - L^q$ estimate as (1.5) in the two space dimensions. To derive the linear estimate (1.5), Nishihara used the explicit form of the fundamental solution to the linear damped wave equation which is expressed by the Bessel function. One-dimensional result can be also dealt with the exact form of the fundamental solutions. However, the explicit form of the evolution kernel in the two-dimensional case is rather complicated. To avoid this complexity, we employ the Fourier transform and observe the detailed asymptotics of the fundamental solution to (1.2) in two-dimensions. The Taylor expansion of the Fourier transform of the evolution operator implies the asymptotics like (1.5). An analogous method was applied by Matsumura [21], Kawashima–Nakao–Ono [16]. The key point is how to obtain the estimate for the high-frequency part of the fundamental solution. The argument due to Marshall–Strauss–Wainger [20] inspires us to employ the modified way of stationary phase method on the expression of the Fourier transform for the radial function. We should note that quite recently, Narazaki [23] also obtained the similar asymptotic decomposition for any higher dimensions. However, his expansion in the high frequency is not explicitly expressed by the solution of the wave equation but some modulated form. This is partially because the extracting the solution of the wave equations from the solution of the damped wave equation as in the form (1.5) is only possible for the lower dimensional cases $n = 1, 2, 3$.

Next, we give the global existence and decay rate of the solutions to the semilinear problem (1.1) with a small initial data. It is observed by Ono [27] that the semi-linear problem has a global solution for the small data when the exponent is super critical $\alpha > 2/n$. Applying the estimate for the linear problem (1.2) in two dimensions, we show the global existence of the solution to problem (1.1) for a wider class of initial data. To reduce the condition on the initial data, we introduce the Besov space $B^1_{2,1}(\mathbb{R}^2)$ which is one of the largest class embedded into L^∞ in $n = 2$. We employ the successive approximation for the corresponding integral equation and obtain the solution by the fixed point theorem.

Finally, we consider the asymptotic behavior of the solutions to the semilinear problem. We prove that the solution to nonlinear damped wave equation (1.1) obtained above can be decomposed into the solutions of related forced heat equation and linear wave equations and it holds a related $L^p - L^q$ type estimate to (1.5) with v being changed to the solution of the inhomogeneous heat equation. As a by-product of this estimate, we show that the solution of nonlinear damped wave equation approximately converges to the heat kernel as the self-similar profile, namely

$$t^{1-1/p}||u(t) - MG_t||_{L^p} = o(1) \tag{1.6}$$

as $t \to \infty$, where M is a certain constant and G_t is the two-dimensional heat kernel. This nature was observed in the study for the semilinear heat equations (cf. Escobedo-Zuazua [7] for heat convection case and Carpio [5], Fujigaki–Miyakawa [8] for the case of the Navier–Stokes equation). For the semilinear damped wave equations, Gallay–Raugel [10] showed (1.6) for $n = 1$ in slightly general equation (see also Karch [15] for the case $4/n < \alpha$, $n = 2, 3$). Our result covers the whole super critical case $\alpha > 1$.

Before closing this section, we give some notations to be used below. Let $\mathcal{F}[f]$ denote the Fourier transform of f defined by

$$\mathcal{F}[f](\xi) := \hat{f}(\xi) := c_n \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$$

with $c_n = (2\pi)^{-\frac{n}{2}}$. For $r > 0$, $B_r(x_0) = \{x \in \mathbb{R}^n: |x - x_0| < r\}$ is the open ball in $\mathbb{R}^n$ and $B_r = B_r(0)$. The norm of a Banach space Z is denoted by $||\cdot||_Z$. For $k \in \mathbb{N} \cup \{0\}$, $p \in [1, \infty]$, $W^{k,p}(\mathbb{R}^n)$ is the Sobolev space;

$$W^{k,p}(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{R}: ||f||_{W^{k,p}} := ||f||_{L^p} + \sum_{|\alpha| \leqslant k} ||\nabla^\alpha f||_{L^p} < \infty \right\}.$$

The Sobolev space introduced by the Fourier transform is written as $H^{s,p}(\mathbb{R}^n)$; for $s \in \mathbb{R}$,

$$H^{s,p}(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{R}: ||f||_{H^{s,p}} := ||\mathcal{F}^{-1}[\langle \xi \rangle^s \hat{f}(\xi)]||_{L^p} < \infty \right\},$$

where $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$. It is well known that the Fourier multiplier theorem implies $W^{k,p}(\mathbb{R}^n) = H^{k,p}(\mathbb{R}^n)$ if $1 < p < \infty$. Let $\{\hat{\phi}_j(\xi)\}_{j=-\infty}^{\infty} \subset C_0^\infty(\mathbb{R}^n_\xi)$ be the Littlewood–Paley dyadic decomposition:

$$\begin{cases} \hat{\psi}(\xi),\ \hat{\phi}(\xi) \geqslant 0: \quad \text{radially symmetric}, \\ \text{supp } \hat{\phi}(\xi) \subset B_2 \backslash B_{\frac{1}{2}}, \quad \text{supp } \hat{\psi}(\xi) \subset B_2, \\ \hat{\phi}_j(\xi) = \hat{\phi}\left(\dfrac{\xi}{2^j}\right), \quad \sum_{j=0}^{\infty} \hat{\phi}_j(\xi) = 1, \quad \text{with } \hat{\phi}_0 := \hat{\psi}. \end{cases}$$

By using the Littlewood–Paley dyadic decomposition, the Besov space $B^s_{p,\sigma}$ is defined as follows:

For $s \geqslant 0$ and $1 \leqslant p,\ \sigma \leqslant \infty$,

$$B^s_{p,\sigma}(\mathbb{R}^n) := \left\{ u : \mathbb{R}^n \to \mathbb{R} \colon ||u||_{B^s_{p,\sigma}} := \left(\sum_{j \geqslant 0} 2^{\sigma j s} ||\phi_j * u||^{\sigma}_{L^p} \right)^{\frac{1}{\sigma}} < \infty \right\}.$$

See for the details of the Besov space [2].

2. Main theorems

Our first theorem is the result for the Cauchy problem of the linear equation

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^2, \\ \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}^2. \end{cases} \tag{2.1}$$

Theorem 2.1. *For $1 \leqslant q \leqslant p \leqslant \infty$ and $u_0,\ u_1 \in L^q(\mathbb{R}^2)$, let $u(t,x)$ be the solutions to* (2.1) *and $v(t,x)$, $w(t,x)$, $\tilde{w}(t,x)$ be the solutions to the following Cauchy problems* (2.2), (2.3), *and* (2.4), *respectively,*

$$\begin{cases} \partial_t v - \Delta v = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ v(0,x) = u_0(x) + u_1(x), & x \in \mathbb{R}^2, \end{cases} \tag{2.2}$$

$$\begin{cases} \partial_t^2 w - \Delta w = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ w(0,x) = u_0(x), & x \in \mathbb{R}^2, \\ \partial_t w(0,x) = u_1(x), & x \in \mathbb{R}^2, \end{cases} \tag{2.3}$$

$$\begin{cases} \partial_t^2 \tilde{w} - \Delta \tilde{w} = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ \tilde{w}(0,x) = 0, & x \in \mathbb{R}^2, \\ \partial_t \tilde{w}(0,x) = u_0(x), & x \in \mathbb{R}^2. \end{cases} \tag{2.4}$$

Then for $\gamma = \frac{1}{q} - \frac{1}{p}$, we have the following $L^p - L^q$ estimate

$$\left|\left| u(t) - v(t) - e^{-\frac{t}{2}} \left\{ w(t) + \left(\frac{1}{2} + \frac{t}{8} \right) \tilde{w}(t) \right\} \right|\right|_{L^p} \leqslant \begin{cases} Ct^{-\gamma-1}(||u_0||_{L^q} + ||u_1||_{L^q}), & t \geqslant 1, \\ Ct^{-\gamma}(||u_0||_{L^q} + ||u_1||_{L^q}), & 0 < t < 1. \end{cases} \tag{2.5}$$

Remark. (i) Let $u(t,x)$ be the solutions to (2.1) and $v(t,x)$ be the solutions to the Cauchy problems (2.2). Narazaki [23] shows the following estimates for $\gamma = \frac{1}{q} - \frac{1}{p}$ with $1 < q \leqslant p < \infty$,

$$||u(t) - v(t) - e^{-\frac{t}{2}}(M_0(t)u_0 + M_1(t)u_1)||_{L^p} \leqslant C(1+t)^{-\frac{n\gamma}{2}-1}(||u_0||_{L^q} + ||u_1||_{L^q}) \tag{2.6}$$

for all $t > 1$, where

$$M_1(t,x) = \mathcal{F}^{-1}\left[\frac{1}{\sqrt{|\xi|^2 - \frac{1}{4}}}\left(\sin t|\xi| \sum_{0 \leqslant k < \frac{n-1}{4}} \frac{(-1)^k}{(2k)!} t^{2k}\, \theta(\xi)^{2k} \right.\right.$$
$$\left.\left. - \cos t|\xi| \sum_{0 \leqslant k < \frac{n-3}{4}} \frac{(-1)^k}{(2k+1)!} t^{2k+1}\, \theta(\xi)^{2k+1}\right)\right]$$

and

$$M_0(t,x) = \mathcal{F}^{-1}\left[\cos t|\xi| \sum_{0 \leqslant k < \frac{n+1}{4}} \frac{(-1)^k}{(2k)!} t^{2k}\, \theta(\xi)^{2k}\right.$$
$$\left. + \sin t|\xi| \sum_{0 \leqslant k < \frac{n-1}{4}} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \theta(\xi)^{2k+1}\right] + \frac{1}{2} M_1(t,x)$$

and $\theta(\xi) \coloneqq |\xi| - \sqrt{|\xi|^2 - \frac{1}{4}}$. The above result shows that when the dimension is higher the terms of $M_0(t,x)$ and $M_1(t,x)$ are increasing. Even when the dimension is two, it can be simplified as

$$\left\| u(t) - v(t) - e^{-\frac{t}{2}} \mathcal{F}^{-1}\left[\cos(t|\xi|)\hat{u_0} + \frac{\sin(t|\xi|)}{\sqrt{|\xi|^2 - \frac{1}{4}}}\hat{u_1}\right.\right.$$
$$\left.\left. + \frac{1}{2}\frac{\sin(t|\xi|)}{\sqrt{|\xi|^2 - \frac{1}{4}}}\hat{u_0} + t\left(|\xi| - \sqrt{|\xi|^2 - \frac{1}{4}}\right)\sin(t|\xi|)\hat{u_0}\right]\right\|_{L^p}$$
$$\leqslant C(1+t)^{-\gamma-1}(||u_0||_{L^q} + ||u_1||_{L^q}) \tag{2.7}$$

for all $t > 1$. As is seen, the terms appearing (2.7) is not exactly the solution of wave equation, while our result is given by the solution of wave equation. Also we remark that our result includes the case $q = 1$, $p = \infty$.

(ii) When $n=2$, it is well known that the solution $v(t,x)$ of (2.2) satisfies the following L^p-L^q estimate;

$$\|v(t)\|_{L^p} \leqslant Ct^{-\gamma}(\|u_0\|_{L^q}+\|u_1\|_{L^q}), \quad 1 \leqslant q \leqslant p \leqslant \infty.$$

Similarly $w(t,x)$ and $\tilde{w}(t,x)$ also satisfy

$$\|w(t)\|_{L^p} \leqslant Ct^{-\gamma'}(\||\nabla|^{\mu}u_0\|_{L^q}+\||\nabla|^{\mu-1}u_1\|_{L^q}),$$

$$\left(2 \leqslant p < \frac{2(n+1)}{n+1-2\mu}, \frac{1}{p}+\frac{1}{q}=1, \gamma'=n\left(1-\frac{2}{p}\right)-\mu\right).$$

Comparing those estimates, Theorem 2.1 shows that the large-time asymptotic profile of the solution to the damped wave equation is converging to the solution of the heat equation. On the other hand, since estimate (2.5) separates the wave part of the solution to the damped wave equation, the regularity of the initial data is not required. In other words, when we show the L^p estimate for the solution to (2.1), the regularity assumption on the data is required which is completely determined by wave part of the solutions $w(t)$ and $\tilde{w}(t)$.

Our next purpose is to consider the Cauchy problem of the semi-linear equation (1.1). First we show the time-global existence theorem as follows.

Theorem 2.2. *Let* $2 \leqslant p \leqslant \infty$ *and* $u_0 \in B^1_{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, $u_1 \in B^0_{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ *and* $\alpha > 1$. *If* $\|u_0,u_1\|_X := \|u_0\|_{B^1_{2,1}}+\|u_0\|_{L^1}+\|u_1\|_{B^0_{2,1}}+\|u_1\|_{L^1}$ *is sufficiently small, then the solution* $u(t,x)$ *to* (1.1) *uniquely exists in* $C([0,\infty);L^2 \cap L^\infty)$ *and satisfies*

$$\|u(t)\|_{L^p} \leqslant C(1+t)^{-(1-\frac{1}{p})}\|u_0,u_1\|_X. \tag{2.8}$$

Moreover, we have

$$u(t) \in C([0,\infty);H^1) \cap C^1([0,\infty);L^2) \tag{2.9}$$

and

$$\|\nabla u(t)\|_{L^2} \leqslant C(1+t)^{-1}\|u_0,u_1\|_X. \tag{2.10}$$

Remark. Suppose that the data $(u_0,u_1) \in (B^1_{2,1}(\mathbb{R}^2) \cap W^{1,1}(\mathbb{R}^2)) \times (L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2))$, then one can extend estimate (2.8) for $1 \leqslant p \leqslant \infty$ as in [25,27]. Note that the condition on the exponent of the nonlinear term is exactly the critical case; $\alpha > 1$. When $\alpha \leqslant 1$, it is known that there exists blow up solution in $n=2$ (see [18,31]).

When $n=2$ and $\alpha > 1$, Matsumura [21] showed the global existence of a small solution to (1.1) with small initial data $(u_0,u_1) \in (H^4(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \times$

$(H^3(\mathbb{R}^2)\cap L^1(\mathbb{R}^2))$ and its decay in $L^2(\mathbb{R}^2)$ and $L^\infty(\mathbb{R}^2)$. The decay order is the same as (2.8) when $p=2$ and $p=\infty$. Nakao–Ono [22] has obtained the global solution $u(t)\in C([0,\infty);H^1)\cap C^1([0,\infty);L^2)$ for $\frac{4}{n}\leqslant\alpha<\frac{4}{n-2}$ $(n\geqslant 3)$, $\frac{4}{n}\leqslant\alpha<\infty$ $(n=1,2)$ and showed the decay with small initial data. When $n=3$, Nishihara [25] obtained the global existence of a unique solution to (1.1) with the condition $(u_0,u_1)\in(W^{1,1}\cap W^{1,\infty})\times(L^1\cap L^\infty)$ and $\alpha>\frac{2}{3}$. It is also showed in [25] that the solution has the L^p-L^q decay property. Ono also proved the similar global result [27]. Although the class of initial data $B^1_{2,1}$ in our Theorem 2 is slightly smaller than the energy class H^1, it is one of the largest class that includes L^∞ and the condition of the derivatives of data is weakest as known.

We give the two-dimensional version for the result of the semilinear equation which corresponds to the result due to Nishihara [25] when $n=3$.

Theorem 2.3. *Let $\alpha>1$, $1\leqslant p\leqslant\infty$, $u_0\in B^1_{2,1}(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$ and $u_1\in B^0_{2,1}(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$. For a solution $u(t,x)$ to* (1.1), *let $\bar{v}(t,x)$ be the solution to the following perturbed heat equation*:

$$\begin{cases}\partial_t\bar{v}-\Delta\bar{v}=|u|^\alpha u, & (t,x)\in(0,\infty)\times\mathbb{R}^2,\\ \bar{v}(0,x)=u_0(x)+u_1(x), & x\in\mathbb{R}^2.\end{cases}\tag{2.11}$$

Then if $\|u_0,u_1\|_X:=\|u_0\|_{B^1_{2,1}}+\|u_0\|_{L^1}+\|u_1\|_{B^0_{2,1}}+\|u_1\|_{L^1}$ is sufficiently small, we have for $t\geqslant 1$,

$$\left\|u(t)-\bar{v}(t)-e^{-\frac{t}{2}}\left\{w(t)+\left(\frac{1}{2}+\frac{t}{8}\right)\tilde{w}(t)\right\}\right\|_{L^p}\leqslant Ct^{-(1-\frac{1}{p})-\varepsilon}\|u_0,u_1\|_X,\tag{2.12}$$

where $\varepsilon=\min\{1,\alpha-(1-\frac{1}{p})\}>0$.

Remark. When $n=3$, Nishihara [25] showed the analogous estimate as (2.12) with the initial data $(u_0,u_1)\in(W^{1,1}\cap W^{1,\infty})\times(L^1\cap L^\infty)$ and $\alpha>\frac{2}{3}$. The reason why the regularity assumption of our result is much reduced than [25] is because we remove the wave part $w(t)$ and $\tilde{w}(t)$ as well as heat part $\bar{v}(t)$ from the solution $u(t)$ of (1.1) (see the remark after Theorem 2.1 on the initial data).

Since the above theorem states that the main term of the solution $u(t)$ is asymptotically expressed by $\bar{v}(t)$, one can derive the asymptotic self-similar profile of the solution as is shown in [10] for the one-dimensional case.

Corollary 2.4. *Let $\alpha>1$, $1\leqslant p\leqslant\infty$ and let u_0 satisfy $u_0\in W^{1,1}(\mathbb{R}^2)$ besides the same conditions in Theorem 2.3. For a solution $u(t,x)$ to* (1.1), *we have the following* *asymptotic self-similar form*: *for* $t\to\infty$,

$$\left\|u(t)-MG_t-e^{-\frac{t}{2}}\left\{w(t)+\left(\frac{1}{2}+\frac{t}{8}\right)\tilde{w}(t)\right\}\right\|_{L^p}=o(t^{-(1-\frac{1}{p})}),\tag{2.13}$$

where G_t *denotes the heat kernel* $G_t(x)=\frac{1}{4\pi t}e^{-|x|^2/4t}$ *and*

$$M=\int_{\mathbb{R}^2}(u_0+u_1)\,dx+\int_0^\infty\int_{\mathbb{R}^2}|u(s)|^\alpha u(s)\,dx\,ds$$

unless $M\neq 0$.

To identify the asymptotic self-similar profile is rather well-known for the solution of semilinear heat equations. For the semilinear damped wave equations, Gallay–Raugel [10] considered a slightly general equation in $n=1$ and showed the self-similar profile upto the second order. Karch [15] also considered the similar asymptotics for more general dissipative equations. His results seems to cover the case $n=2,3$ and $4/n<\alpha<4/(n-2)$ for the negative sign of the nonlinear term.

3. Preliminaries

3.1. Interpolation and useful inequalities

In this section, we prepare several lemmas for the following sections. For $f\in L^2\cap L^p$, $1\leqslant p\leqslant\infty$, let $m(\xi)$ be the Fourier multiplier and T_m be the Fourier multiplier operator defined by

$$T_mf\coloneqq\mathcal{F}^{-1}[m\hat{f}](x)=c_n\int_{\mathbb{R}^n}e^{ix\cdot\xi}m(\xi)\hat{f}(\xi)\,d\xi.$$

We define M_p and $M_p(m)$ as follows.

$$M_p\coloneqq\{m:\ \text{there exists } A_p>0 \text{ such that } ||T_mf||_{L^p}\leqslant A_p||f||_{L^p}\},$$

$$M_p(m)\coloneqq\sup_f\frac{||T_mf||_{L^p}}{||f||_{L^p}},\quad m\in M_p.$$

The following lemma is well known Carleson–Beurling's inequality and is useful to show the L^p boundedness for Fourier multipliers.

Lemma 3.1 (Carleson–Beurling's inequality). *If* $m \in B^{\frac{n}{2}}_{2,1}(\mathbb{R}^n_\xi)$, *then* $m \in M_\infty$ *and the inequality*

$$M_\infty(m) \leqslant C \|m\|_{L^2}^{1-\frac{n}{2s}} \|m\|_{B^{\frac{n}{2}}_{2,1}}$$

holds. Namely $m \in M_p$ *for all* $p \in [1, \infty]$. *In particular, we have for* $m \in H^s(\mathbb{R}^n_\xi)$ *with* $s > \frac{n}{2}$,

$$M_\infty(m) \leqslant C \|m\|_{L^2}^{1-\frac{n}{2s}} \||\nabla|^s m\|_{L^2}^{\frac{n}{2s}}.$$

See for the proof of Lemma 3.1 [4, p. 18].

We also use the following well-known interpolation theorem (see [2]).

Lemma 3.2 (The Riesz–Thorin complex interpolation). *For* $1 \leqslant p_0, p_1 \leqslant \infty$, *let* T *be the linear operator bounded from* L^{q_0} *to* L^{p_0} *and* L^{q_1} *to* L^{p_1}; *namely for some* $M_0, M_1 > 0$,

$$\|Tf\|_{L^{p_0}} \leqslant M_0 \|f\|_{L^{q_0}},$$

$$\|Tf\|_{L^{p_1}} \leqslant M_1 \|f\|_{L^{q_1}}.$$

Then for $\theta \in (0,1)$ *andf* $\in L^q$, *we have*

$$\|Tf\|_{L^p} \leqslant M_0^{1-\theta} M_1^{\theta} \|f\|_{L^q},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

The following lemma is useful to estimate the decay of the solution.

Lemma 3.3. *Let* $a > 0$, $b > 0$ *and* $\max(a,b) > 1$. *There exist a constant* C *depending only on* a *and* b *such that the following inequality holds*:

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}\, ds \leqslant C(1+t)^{-\min(a,b)}.$$

In particular, we have

$$\int_0^t e^{-c(t-s)}(1+s)^{-b}\, ds \leqslant C(1+t)^{-b}$$

for any $c > 0$.

See for the proof of Lemma 3.3 [28].

3.2. Solutions of linear equation

According to Courant–Hilbert [6], we introduce the Fourier transform of the fundamental solution to the damped wave equation (cf. [21]). Let $K_0(t)$, $K_1(t)$ be

$$K_0(t) := e^{-\frac{t}{2}} \cos\{t\alpha(|\nabla|)\}, \tag{3.1}$$

$$K_1(t) := e^{-\frac{t}{2}} \frac{\sin\{t\alpha(|\nabla|)\}}{\alpha(|\nabla|)}, \tag{3.2}$$

where

$$\mathcal{F}[\alpha(|\nabla|)](\xi) = \alpha(\xi) = \begin{cases} \sqrt{|\xi|^2 - \dfrac{1}{4}}, & |\xi| > \dfrac{1}{2}, \\ i\sqrt{\dfrac{1}{4} - |\xi|^2}, & |\xi| \leqslant \dfrac{1}{2}. \end{cases}$$

Then the solution $u(t)$ of linear equation (2.1) is given through the Fourier transform by $K_0(t)$ and $K_1(t)$ as

$$u(t,x) = K(u_0, u_1) := K_0(t)u_0 + K_1(t)\left(\frac{1}{2}u_0 + u_1\right). \tag{3.3}$$

The Duhamel principle implies that the solution $u(t)$ of nonlinear equation (1.1) solves the integral equation

$$u(t,x) = K(u_0, u_1) + \int_0^t K_1(t-s) * |u|^\alpha u(s,\cdot)\, ds. \tag{3.4}$$

Similarly, we introduce the evolution group of the wave equation; let $W_0(t)$, $W_1(t)$ be

$$W_0(t) := \cos(t|\nabla|) = \mathcal{F}^{-1}[\cos(t|\xi|)], \tag{3.5}$$

$$W_1(t) := \frac{\sin(t|\nabla|)}{|\nabla|} = \mathcal{F}^{-1}\left[\frac{\sin(t|\xi|)}{|\xi|}\right]. \tag{3.6}$$

The solution $w(t)$ of linear wave equation (2.3) is also given by

$$w(t) := W_0(t)u_0 + W_1(t)u_1. \tag{3.7}$$

One of the evolution operators $W_1(t)$ satisfies the following estimate:

Lemma 3.4. (i) *Let* $t \geqslant 0$, $2 \leqslant p \leqslant \frac{2(n+1)}{n-1}$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then the estimate*

$$||W_1(t)g||_{L^p} \leqslant Ct^{n(1-\frac{2}{p})-1}||g||_{L^{p'}} \tag{3.8}$$

holds for $g \in L^{p'}(\mathbb{R}^n)$.

(ii) *When* $n = 2$, $W_1(t)$ *satisfies the following* $L^\infty - L^\infty$ *estimate*

$$||W_1(t)g||_{L^\infty} \leqslant t||g||_{L^\infty} \tag{3.9}$$

for $g \in L^\infty(\mathbb{R}^2)$. *Especially, the operator* $W_1(t)$ *belongs to the class* M_p *for all* $1 \leqslant p \leqslant \infty$ *with the bound* t.

Proof of Lemma 3.4. (i) See for example [3,11,30].

(ii) When $n = 2$, the fundamental solution $W_1(t)$ is given as follow (see [6,26]):

$$W_1(t)g = \frac{1}{2\pi}\int\int_{|x-y|\leqslant t} \frac{g(y)}{\sqrt{t^2 - |x-y|^2}}\,dy.$$

Thus

$$\begin{aligned}
\left|\frac{1}{2\pi}\int_{|x-y|\leqslant t} \frac{g(y)}{\sqrt{t^2-|x-y|^2}}\,dy\right| &= \frac{t}{2\pi}\left|\int_{|\frac{x}{t}-z|\leqslant 1} \frac{g(tz)}{\sqrt{1-|\frac{x}{t}-z|^2}}\,dz\right| \\
&\leqslant \frac{t}{2\pi}\sup_{|x-y|\leqslant t}|g(y)|\int_{|\frac{x}{t}-z|\leqslant 1}\frac{1}{\sqrt{1-|\frac{x}{t}-z|^2}}\,dz \\
&\leqslant t||g||_\infty \int_0^1 \frac{r}{\sqrt{1-r^2}}\,dr = t||g||_\infty. \qquad \square
\end{aligned}$$

4. Proof of Theorem 2.1

We show that the solution of the linear damped wave equation can be approximated by the solutions of heat and wave equations. This can be observed by a heuristic argument as follows. By the definition of $\widehat{K}_0(t)$, $\widehat{K}_1(t)$, we see for the low frequency part $|\xi| \leqslant \frac{1}{2}$ that

$$\begin{aligned}
\widehat{K}_0(t) &= e^{-\frac{t}{2}}\cosh\left(t\sqrt{\frac{1}{4}-|\xi|^2}\right) \\
&= \frac{1}{2}\left(e^{t\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-|\xi|^2}\right)} + e^{t\left(-\frac{1}{2}-\sqrt{\frac{1}{4}-|\xi|^2}\right)}\right),
\end{aligned} \tag{4.1}$$

$$\begin{aligned}\widehat{K}_1(t) &= e^{-\frac{t}{2}} \frac{\sinh\left(t\sqrt{\frac{1}{4}-|\xi|^2}\right)}{\sqrt{\frac{1}{4}-|\xi|^2}} \\ &= \frac{1}{\sqrt{1-4|\xi|^2}}\left(e^{t\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-|\xi|^2}\right)} - e^{t(-\frac{1}{2}-\sqrt{\frac{1}{4}-|\xi|^2})}\right). \end{aligned} \tag{4.2}$$

The second term of (4.1) and (4.2) decay exponentially as $t \to \infty$. Since $-\frac{1}{2}+\sqrt{\frac{1}{4}-|\xi|^2} \simeq -|\xi|^2$ ($|\xi| \ll 1$), one can regard then as the heat kernel;

$$\widehat{K}_0(t) \simeq \frac{1}{2} e^{-t|\xi|^2}, \quad \widehat{K}_1(t) \simeq e^{-t|\xi|^2}.$$

On the other hand, for the high-frequency part $|\xi| > 1$,

$$\begin{aligned}\widehat{K}_0(t) = e^{-\frac{t}{2}} \cos\{t\alpha(\xi)\} &\simeq e^{-\frac{t}{2}} \cos(t|\xi|) + e^{-\frac{t}{2}} \frac{t}{8|\xi|} \sin(t|\xi|) + e^{-\frac{t}{2}} O\left(\frac{t^\delta}{|\xi|^2}\right) \\ &= e^{-\frac{t}{2}} W_0(t) + e^{-\frac{t}{2}} \frac{t}{8} W_1(t) + e^{-\frac{t}{2}} O\left(\frac{t^\delta}{|\xi|^2}\right), \end{aligned} \tag{4.3}$$

$$\begin{aligned}\widehat{K}_1(t) = e^{-\frac{t}{2}} \frac{\sin\{t\alpha(\xi)\}}{\alpha(\xi)} &\simeq e^{-\frac{t}{2}} \frac{\sin(t|\xi|)}{|\xi|} + e^{-\frac{t}{2}} O\left(\frac{t^\delta}{|\xi|^2}\right) \\ &= e^{-\frac{t}{2}} W_1(t) + e^{-\frac{t}{2}} O\left(\frac{t^\delta}{|\xi|^2}\right), \end{aligned} \tag{4.4}$$

where $\delta > 0$. Hence one can represent the error of the solution as

$$\begin{aligned} &\left\{K_0(t) - e^{-\frac{t}{2}}\left(W_0(t) + \frac{t}{8} W_1(t)\right)\right\} u_0 - \frac{1}{2} e^{t\Delta} u_0 \\ &\quad + (K_1(t) - e^{-\frac{t}{2}} W_1(t))\left(\frac{1}{2} u_0 + u_1\right) - e^{t\Delta}\left(\frac{1}{2} u_0 + u_1\right) \\ &= K_0(t) u_0 + K_1(t)\left(\frac{1}{2} u_0 + u_1\right) - e^{t\Delta}(u_0 + u_1) \\ &\quad - e^{-\frac{t}{2}}\left\{ W_0(t) u_0 + W_1(t) u_1 + \left(\frac{1}{2} + \frac{t}{8}\right) W_1(t) u_0 \right\} \\ &= u(t,x) - v(t,x) - e^{-\frac{t}{2}}\left\{ w(t) + \left(\frac{1}{2} + \frac{t}{8}\right) \tilde{w}(t) \right\}. \end{aligned} \tag{4.5}$$

According to the above observation, to prove Theorem 1, it suffices to show the following proposition.

Proposition 4.1. *Let* $K_0(t)$, $K_1(t)$ *be the evolution operator of* (2.1) *and* $W_0(t)$, $W_1(t)$ *be defined in* (3.5) *and* (3.6). *Then for* $1 \leqslant q \leqslant p \leqslant \infty$ *and* $g \in L^q(\mathbb{R}^2)$, *we have*

$$\left\| \left\{ K_0(t) - e^{-\frac{t}{2}} \left(W_0(t) + \frac{t}{8} W_1(t) \right) - \frac{1}{2} e^{t\Delta} \right\} g \right\|_{L^p} \leqslant \begin{cases} Ct^{-\gamma-1} ||g||_{L^q}, & t \geqslant 1, \\ Ct^{-\gamma} ||g||_{L^q}, & 0 < t < 1, \end{cases} \tag{4.6}$$

$$||(K_1(t) - e^{-\frac{t}{2}} W_1(t))g - e^{t\Delta} g||_{L^p} \leqslant \begin{cases} Ct^{-\gamma-1} ||g||_{L^q}, & t \geqslant 1, \\ Ct^{-\gamma} ||g||_{L^q}, & 0 < t < 1, \end{cases} \tag{4.7}$$

where $\gamma = \frac{1}{q} - \frac{1}{p}$.

We introduce a cut-off function into the high- and low-frequency parts. Let χ_h, $\chi_l \in C^\infty$ be

$$\chi_h(\xi) = \begin{cases} 1, & |\xi| > 2, \\ 0, & |\xi| \leqslant 1, \end{cases} \tag{4.8}$$

$$\chi_l(\xi) = \begin{cases} 0, & |\xi| > \frac{1}{3}, \\ 1, & |\xi| \leqslant \frac{1}{4}. \end{cases} \tag{4.9}$$

Then the proof of Proposition 4.1 can be reduced into the following proposition.

Proposition 4.2. *Let* $1 \leqslant q \leqslant p \leqslant \infty$, $g \in L^q(\mathbb{R}^2)$ *and* χ_h, χ_l *be cut-off functions defined by* (4.8) *and* (4.9). *Then for all* $t \geqslant 0$, *the following estimates hold with positive constant* δ *and* $\gamma = \frac{1}{q} - \frac{1}{p}$:

$$\left\| \check{\chi}_h * \left\{ K_0(t) - e^{-\frac{t}{2}} \left(W_0(t) + \frac{t}{8} W_1(t) \right) - \frac{1}{2} e^{t\Delta} \right\} g \right\|_{L^p} \leqslant Ct^{-\gamma} e^{-\delta t} ||g||_{L^q}. \tag{4.10}$$

$$||\check{\chi}_h * \left\{ \left(K_1(t) - e^{-\frac{t}{2}} W_1(t) \right) - e^{t\Delta} \right\} g||_{L^p} \leqslant Ct^{-\gamma} e^{-\delta t} ||g||_{L^q}. \tag{4.11}$$

$$\left\| \check{\chi}_l * \left\{ K_0(t) - e^{-\frac{t}{2}} \left(W_0(t) + \frac{t}{8} W_1(t) \right) - \frac{1}{2} e^{t\Delta} \right\} g \right\|_{L^p} \leqslant C(1+t)^{-\gamma-1} ||g||_{L^q}. \tag{4.12}$$

$$||\check{\chi}_l * \left\{ (K_1(t) - e^{-\frac{t}{2}} W_1(t)) - e^{t\Delta} \right\} g||_{L^p} \leqslant C(1+t)^{-\gamma-1} ||g||_{L^q}. \tag{4.13}$$

Once Proposition 4.2 is proved, one can show Proposition 4.1 by noticing the following obvious estimates for the middle-frequency parts. Let $\chi_m(\xi) := 1 - \chi_h(\xi) - \chi_l(\xi)$ then we have for $1 \leqslant q \leqslant p \leqslant \infty$, $u_0 \in L^q(\mathbb{R}^2)$ and $u_1 \in L^q(\mathbb{R}^2)$,

$$\left\| \mathcal{F}^{-1}\left[\chi_m\left\{K_0(t) - e^{-\frac{t}{2}}\left(W_0(t) + \frac{t}{8}W_1(t)\right) - \frac{1}{2}e^{t\Delta}\right\}g\right]\right\|_{L^p} \leqslant Ce^{-\delta t}||g||_{L^q}, \quad t>0 \quad (4.14)$$

$$||\mathcal{F}^{-1}[\chi_m\{(K_1(t) - e^{-\frac{t}{2}}W_1(t)) - e^{t\Delta}\}g]||_{L^p} \leqslant Ce^{-\delta t}||g||_{L^q}, \quad t>0 \quad (4.15)$$

for some $0<\delta<1/2$. Since the symbols are compactly supported, bounded and smooth around $|\xi| = \frac{1}{2}$, the estimates (4.14) and (4.15) are easily follows from the Hausdorff–Young inequality.

Proof of Proposition 4.2. We show only (4.10) and (4.12). The other cases (4.11) and (4.13) are slightly simpler and one can show them in a similar way.

Let us consider the two Fourier multipliers;

$$m_h(\xi) := \chi_h(\xi)\left\{e^{-\frac{t}{2}}\left(\cos\{t\alpha(\xi)\} - \cos(t|\xi|) - \frac{t}{8}\frac{\sin(t|\xi|)}{|\xi|}\right) - \frac{1}{2}e^{-t|\xi|^2}\right\}, \quad (4.16)$$

$$m_l(\xi) := \chi_l(\xi)\left\{e^{-\frac{t}{2}}\left(\cos\{t\alpha(\xi)\} - \cos(t|\xi|) - \frac{t}{8}\frac{\sin(t|\xi|)}{|\xi|}\right) - \frac{1}{2}e^{-t|\xi|^2}\right\}. \quad (4.17)$$

We first claim that by Lemma 3.1, $m_h(\xi)$, $m_l(\xi) \in M_p$ for all $1 \leqslant p \leqslant \infty$. This shows that the corresponding Fourier multiplier operators T_{m_h} and T_{m_l} are L^p bounded. Namely, we show the estimate of (4.10) and (4.12) for $g \in L^p$ with $1 \leqslant q = p \leqslant \infty$:

$$||T_{m_h}g||_{L^p} \leqslant C(1+t)^2 e^{-t/2}||g||_{L^p}, \quad (4.18)$$

$$||T_{m_l}g||_{L^p} \leqslant C(1+t)^{-1}||g||_{L^p}. \quad (4.19)$$

Next we claim the $L^\infty - L^1$ estimates for T_{m_h} and T_{m_l}: For $g \in L^1$,

$$||T_{m_h}g||_{L^\infty} \leqslant Ct^{-1}e^{-\delta t}||g||_{L^1}, \quad (4.20)$$

$$||T_{m_l}g||_{L^\infty} \leqslant C(1+t)^{-2}||g||_{L^1}, \quad (4.21)$$

then combining those estimates, Lemma 3.2 (the Riesz–Thorin complex interpolation theorem) yields

$$||T_{m_h}g||_{L^p} \leqslant Ct^{-\gamma}e^{-\delta t}||g||_{L^q}, \quad (4.22)$$

$$||T_{m_l}g||_{L^p} \leqslant C(1+t)^{-\gamma-1}||g||_{L^q}. \quad (4.23)$$

for $1 \leqslant q \leqslant p \leqslant \infty$, $\gamma = \frac{1}{q} - \frac{1}{p}$ which is the desired estimate (4.10) and (4.12).

We firstly claim that $m_h \in M_p$ for all $p \in [1, \infty]$, namely show that (4.18). Since the heat evolution is uniformly L^p-bounded by L^1 summability of its kernel, we show the bound for the oscillating part. Since $\alpha(\xi) - |\xi| = -\frac{1}{8|\xi|} + O(\frac{1}{|\xi|^3})$ for $|\xi| > 1$, the mean value theorem yields for some $\theta, \theta' \in (0, 1)$,

$$\begin{aligned}
f(t,\xi) &:= \cos\{t\alpha(\xi)\} - \cos(t|\xi|) - \frac{t}{8|\xi|}\sin(t|\xi|) \\
&= \frac{t}{8|\xi|}\sin\{t(|\xi| + \theta(\alpha(\xi) - |\xi|))\} - \frac{t}{8|\xi|}\sin(t|\xi|) + A(t,\xi) \\
&= \frac{t^2}{8|\xi|}\theta(\alpha(\xi) - |\xi|)\cos\{t(|\xi| + \theta'(\alpha(\xi) - |\xi|))\} + A(t,\xi) \\
&= -\frac{t^2}{64|\xi|^2}\theta\cos\{t(|\xi| + \theta'(\alpha(\xi) - |\xi|))\} + A(t,\xi) + B(t,\xi) \\
&= -\frac{t^2}{64}\theta\frac{\cos(t|\xi|)}{|\xi|^2} + \frac{t^2}{64}\theta\frac{1}{|\xi|^2}\{\cos\{t(|\xi| + \theta'(\alpha(\xi) - |\xi|))\} - \cos t|\xi|\} \\
&\quad + A(t,\xi) + B(t,\xi), \qquad (4.24)
\end{aligned}$$

where

$$A(t,\xi) := -t\left(\alpha(\xi) - |\xi| + \frac{1}{8|\xi|}\right)\sin\{t(|\xi| + \theta(\alpha(\xi) - |\xi|))\} = O\left(\frac{t}{|\xi|^3}\right),$$

$$B(t,\xi) := \frac{\theta t^2}{8|\xi|}\left(\alpha(\xi) - |\xi| + \frac{1}{8|\xi|}\right)\cos\{t(|\xi| + \theta'(\alpha(\xi) - |\xi|))\} = O\left(\frac{t^2}{|\xi|^4}\right) \qquad (4.25)$$

for $|\xi| > 1$ and $t > 0$. Since the right-hand side of (4.24) is order $O(|\xi|^{-2})$ and the symbol $\chi_h f$ is smooth and supported over $|\xi| > 1$, we have $\|\chi_h f\|_{L^2} \leqslant C(1+t)^2 e^{-t/2}$. Analogously, the derivatives up to the second order of the right-hand side of (4.24) still have the same order $O(|\xi|^{-2})$ when $|\xi| \gg 1$. This shows that $\nabla^2(\chi_h f) \in L^2$ with $\|\nabla^2(\chi_h f)\|_{L^2} \leqslant C(1+t)^4 e^{-t/2}$. Hence Lemma 3.1 implies (4.18).

Next we show (4.19). To see it, we need to be a little more careful for the order of the time decay. Observing expression (4.1), we set

$$g(t,\xi) := \chi_l(\xi)\left(e^{-\frac{t}{2}}\cosh\{i^{-1}t\alpha(\xi)\} - \frac{1}{2}e^{-t|\xi|^2}\right).$$

Then the multiplier can be written as

$$m_l(\xi) = g(t,\xi) - \chi_l(\xi)\left(e^{-\frac{t}{2}}\left(\cos(t|\xi|) - \frac{t}{8}\frac{\sin(t|\xi|)}{|\xi|}\right)\right). \qquad (4.26)$$

The second part of the right-hand side of (4.26) is compactly supported and decays exponentially. Hence, its L^2 norm is bounded by a constant with $e^{-t/2}$ factor. Let $\phi_j(\xi)$ be the Littlewood–Paley decomposition in ξ variable. Then since

$$\sum_{j=0}^{\infty} 2^j \|\phi_j * (\chi_l(\xi)\cos(t|\xi|))\|_{L^2} \leqslant \|\psi * (\chi_l(\xi)\cos(t|\xi|))\|_{L^2} \leqslant C, \tag{4.27}$$

we have $\|\chi_l(\xi)\cos(t|\xi|)\|_{B^1_{2,1}(\mathbb{R}^2_\xi)} \leqslant Ct$ and the first assertion of Lemma 3.1 yields the L^p boundedness of the cosine part of the right-hand side of (4.26) with the bound $Cte^{-t/2}$. For the sine part, we note that multiplying χ_l is a smoothing operator and is certainly L^p bounded. The oscillation part can be bound by (3.9) in Lemma 3.4. Hence, we have the L^p boundedness for the Fourier multiplier of the second part of (4.26). The first part can be treated as

$$\begin{aligned} g(t,\xi) &= \frac{1}{2}\chi_l(\xi)\left\{ e^{-t|\xi|^2}\left(e^{-\frac{t}{2}\left(1-\sqrt{1-4|\xi|^2}-2|\xi|^2\right)} - 1\right) + e^{-\frac{t}{2}\left(1+\sqrt{1-4|\xi|^2}\right)}\right\} \\ &= \frac{1}{2}\chi_l(\xi)h(t,\xi) + O(e^{-t}), \end{aligned} \tag{4.28}$$

where

$$h(t,\xi) := e^{-t|\xi|^2}\left(e^{-\frac{t}{2}(1-\sqrt{1-4|\xi|^2}-2|\xi|^2)} - 1\right).$$

Again the second part of the $g(t,\xi)$ has a compact support and decays exponentially. Hence the issue is reduced to $h(t,\xi)$. Since $h(t,\xi)$ is the radially symmetric function, the second derivative with respect to ξ is

$$\begin{aligned} \partial^2_{|\xi|} h(t,\xi) = &- 2te^{-t|\xi|^2}\left(\frac{1}{\sqrt{1-4|\xi|^2}} e^{-\frac{t}{2}(1-\sqrt{1-4|\xi|^2}-2|\xi|^2)} - 1\right) \\ &+ 4t^2\xi^2 e^{-t|\xi|^2}\left(\frac{1}{\sqrt{1-4|\xi|^2}} e^{-\frac{t}{2}(1-\sqrt{1-4|\xi|^2}-2|\xi|^2)} - 1\right) \\ &- 2t\xi e^{-t|\xi|^2}\frac{d}{d\xi}\left(\frac{1}{\sqrt{1-4|\xi|^2}} e^{-\frac{t}{2}(1-\sqrt{1-4|\xi|^2}-2|\xi|^2)}\right) \\ =&: J_1 + J_2 + J_3, \end{aligned} \tag{4.29}$$

where $|\xi| \leqslant \frac{1}{4}$. Since $|e^{-\frac{t}{2}(1-\sqrt{1-4|\xi|^2}-2|\xi|^2)} - 1| \leqslant 4t|\xi|^4$ for $0 \leqslant |\xi| \leqslant \frac{1}{4}$, using

$$\int_{|\xi| \leqslant 1/4} |\xi|^k e^{-t|\xi|^2}\, d\xi \leqslant C(1+t)^{-\frac{k+2}{2}} \quad \text{for all } t \geqslant 0, \tag{4.30}$$

we have

$$\|\chi_l J_1\|_{L^2}^2 \leqslant 64t^4 \int_{|\xi| \leqslant \frac{1}{4}} |\xi|^8 e^{-2t|\xi|^2}\, d\xi \leqslant C(1+t)^{-1}. \tag{4.31}$$

Thus (4.31) gives

$$\|\chi_l J_1\|_{L^2(\mathbb{R}^2_\xi)} \leqslant C(1+t)^{-\frac{1}{2}} \tag{4.32}$$

and similarly

$$\|\chi_l J_2\|_{L^2(\mathbb{R}^2_\xi)} \leqslant C(1+t)^{-\frac{1}{2}}. \tag{4.33}$$

Observing

$$\begin{aligned}
&\frac{d}{d\xi}\left(\frac{1}{\sqrt{1-4|\xi|^2}} e^{-\frac{t}{2}(1-\sqrt{1-4|\xi|^2}-2|\xi|^2)}\right) \\
&\quad = \frac{4\xi}{(1-4|\xi|^2)^{\frac{3}{2}}}\left(e^{-\frac{t}{2}(1-\sqrt{1-4|\xi|^2}-2|\xi|^2)} - 1\right) + \frac{4\xi}{(1-4|\xi|^2)^{\frac{3}{2}}} \\
&\quad\quad - \tfrac{2t\xi}{1-4|\xi|^2}\left(1-\sqrt{1-4|\xi|^2}\right) e^{-\frac{t}{2}\left(1-\sqrt{1-4|\xi|^2}-2|\xi|^2\right)}.
\end{aligned} \tag{4.34}$$

We obtain for J_3 that by a similar way to (4.30) and (4.31),

$$\|\chi_l J_3\|_{L^2(\mathbb{R}^2_\xi)} \leqslant C(1+t)^{-\frac{1}{2}}. \tag{4.35}$$

Note that the estimates for $\|h\nabla^2 \chi_l\|_{L^2(\mathbb{R}^2_\xi)}$ and $\|\nabla h \nabla \chi_l\|_{L^2(\mathbb{R}^2_\xi)}$ have better decay estimates for t. Hence by (4.32), (4.33) and (4.35), we obtain

$$\|m_l(\xi)\|_{L^2(\mathbb{R}^2_\xi)} \leqslant C(1+t)^{-\frac{3}{2}}, \tag{4.36}$$

$$\|\nabla^2 m_l(\xi)\|_{L^2(\mathbb{R}^2_\xi)} \leqslant C(1+t)^{-\frac{1}{2}}. \tag{4.37}$$

Thus by (4.37), (4.36) and Lemma 3.1, taking $n=2$ and $s=2$, we see

$$M_\infty(m_l) \leqslant C(1+t)^{-1}, \tag{4.38}$$

which proves (4.19).

Next we show (4.20) and (4.21). In view of expansion (4.24), the following estimate is required to derive the high-frequency estimate (4.20) (cf. [20]).

Lemma 4.3. *We have for all $t \geqslant 0$ and $x \in \mathbb{R}^2$,*

$$\left| \mathcal{F}^{-1}\left[\chi_h(\xi)\left(\frac{\cos(t|\xi|)}{|\xi|^2} \right) \right] \right| \leqslant C t^{-1/2}. \tag{4.39}$$

Proof. We use the well-known formula for the Fourier transform of a radial function (cf. [29, p. 155])

$$\mathcal{F}^{-1}\left[\chi_h(\xi)\left(\frac{\cos(t|\xi|)}{|\xi|^2} \right) \right](x) = \int_0^\infty \chi_h(r) r^{-1} \cos(tr) J_0(|x|r)\, dr,$$

where $r = |\xi|$ and J_0 denotes the Bessel function of order 0. The Bessel function J_0 satisfies the following properties (cf. [29, p. 158]).

$$\frac{d}{ds}(J_0(s)) = -J_1(s), \tag{4.40}$$

$$|J_0(s)|,\ |J_1(s)| \leqslant C|s|^{-\frac{1}{2}} \quad \text{for all } s > 0, \tag{4.41}$$

$$J_0(s) = \sqrt{\frac{2}{\pi s}} \cos\left(s - \frac{\pi}{4} \right) + O(s^{-\frac{3}{2}}) \quad \text{as } s \to +\infty, \tag{4.42}$$

$$|J_0(s)| \leqslant 1 \quad \text{for all } s > 0. \tag{4.43}$$

By property (4.42), we have for $|x| > t^{\frac{1}{2}}$

$$\begin{aligned} &\int_0^\infty \chi_h(r) r^{-1} \cos(tr) J_0(|x|r)\, dr \\ &= C|x|^{-\frac{1}{2}} \int_1^\infty \chi_h(r) r^{-\frac{3}{2}} \cos(tr) \cos\left(|x|r - \frac{\pi}{4} \right) dr + O\left(|x|^{-\frac{3}{2}} \int_1^\infty r^{-\frac{5}{2}}\, dr \right). \end{aligned} \tag{4.44}$$

The second term of the right-hand side in (4.44) can be estimate as $O(t^{-1})$ since $|x| > t^{\frac{2}{3}}$. We also estimate the first term by the same way to obtain

$$\begin{aligned}
&|x|^{-\frac{1}{2}}\left|\int_1^\infty \chi_h(r)r^{-\frac{3}{2}}\cos(tr)\cos\left(|x|r-\frac{\pi}{4}\right)dr\right| \\
&\leqslant |x|^{-\frac{1}{2}}\int_1^\infty r^{-\frac{3}{2}}\,dr \\
&\leqslant Ct^{-\frac{1}{2}}.
\end{aligned}$$

For $|x| \leqslant t^{\frac{1}{3}}$, using property (4.40), we integrate by parts to see

$$\begin{aligned}
&\int_0^\infty \chi_h(r)r^{-1}\cos(tr)J_0(|x|r)\,dr \\
&= \int_0^\infty \chi_h(r)(tr)^{-1}\frac{d}{dr}(\sin(tr))J_0(|x|r)\,dr \\
&= -\int_1^2 \chi_h{}'(r)\frac{\sin(tr)}{tr}J_0(|x|r)\,dr \\
&\quad + \int_1^\infty \chi_h(r)\frac{\sin(tr)}{tr}J_0(|x|r)\frac{dr}{r} - |x|\int_1^\infty \chi_h(r)\frac{\sin(tr)}{tr}J_1(|x|r)\,dr. \qquad (4.45)
\end{aligned}$$

The first term can be bounded by a constant for small $t<1$ by (4.43) and bounded by t^{-1} for large t. For the second term we see by (4.43) again that

$$\begin{aligned}
\left|\int_1^\infty \chi_h(r)\frac{\sin(tr)}{tr}J_0(|x|r)\frac{dr}{r}\right| &\leqslant \int_1^\infty \left|\frac{\sin(tr)}{tr}J_0(|x|r)\right|\frac{dr}{r} \\
&\leqslant \int_t^\infty \left|\frac{\sin(r')}{r'}\right|\frac{dr'}{r'} \\
&\leqslant \begin{cases} C\log t, & t<1, \\ Ct^{-1}, & t>1. \end{cases} \qquad (4.46)
\end{aligned}$$

Applying property (4.41), the third term in (4.45) can be bounded by

$$\begin{aligned}
&|x|\left|\int_1^\infty \chi_h(r)\frac{\sin(tr)}{tr}J_1(|x|r)\,dr\right| \\
&\leqslant |x|\int_1^\infty \left|\frac{\sin(tr)}{tr}\right||x|^{-\frac{1}{2}}r^{-\frac{1}{2}}\,dr \\
&\leqslant t^{-1}|x|^{\frac{1}{2}}\int_1^\infty r^{-\frac{3}{2}}\,dr \\
&\leqslant Ct^{-\frac{1}{2}}.
\end{aligned}$$

under $|x|<t$. Hence we showed (4.39). $\square$

Proof of Proposition 4.2 continue. To see that (4.20) holds, it suffices to prove that by Hausdorff–Young's inequality for $t>0$

$$||\mathcal{F}^{-1}[m_h(\xi)]||_{L^\infty} \leqslant Ct^{-1}e^{-\delta t},$$

where $0<\delta<\frac{1}{2}$.

From (4.24), it follows that for $0<\theta<1$ and $|\xi|>1$,

$$\cos\{t\alpha(\xi)\} - \cos(t|\xi|) - \frac{t}{8|\xi|}\sin(t|\xi|) + \frac{t^2}{64}\theta\frac{\cos(t|\xi|)}{|\xi|^2} = A(t,\xi) + B(t,\xi) + C(t,\xi),$$

where

$$C(t,\xi) := \frac{t^2}{64}\theta\frac{1}{|\xi|^2}\{\cos\{t(|\xi| + \theta'(\alpha(\xi) - |\xi|))\} - \cos t|\xi|\} = O\left(\frac{t^3}{|\xi|^3}\right). \tag{4.47}$$

By the Hausdorff–Young inequality for the Fourier transform, we have from the error estimates (4.25) and (4.47) that

$$\begin{aligned} e^{-\frac{t}{2}}&\left|\left|\mathcal{F}^{-1}\left[\chi_h(\xi)\left\{\cos\{t\alpha(\xi)\} - \cos(t|\xi|) - \frac{t}{8}\frac{\sin(t|\xi|)}{|\xi|} - \theta\frac{t^2}{64}\frac{\cos(t|\xi|)}{|\xi|^2}\right\}\right]\right|\right|_{L^\infty} \\ &\leqslant e^{-\frac{t}{2}}\left|\left|\chi_h(\xi)\left\{\cos\{t\alpha(\xi)\} - \cos(t|\xi|) - \frac{t}{8}\frac{\sin(t|\xi|)}{|\xi|} - \theta\frac{t^2}{64}\frac{\cos(t|\xi|)}{|\xi|^2}\right\}\right|\right|_{L^1} \leqslant Ce^{-\delta t}. \end{aligned}$$

From

$$\int_{|\xi|>1} |\xi|^k e^{-t|\xi|^2}\,d\xi \leqslant C_k t^{-\frac{k+2}{2}}e^{-t} \quad \text{for all } t\geqslant 0, \tag{4.48}$$

$$\begin{aligned} ||\mathcal{F}^{-1}[m_h]||_{L^\infty} &\leqslant Ce^{-\delta t} + \left|\left|\chi_h(\xi)\left(\frac{1}{2}e^{-t|\xi|^2}\right)\right|\right|_{L^1} + Ct^2e^{-\frac{t}{2}}\left|\left|\mathcal{F}^{-1}\left[\chi_h(\xi)\left(\frac{\cos(t|\xi|)}{|\xi|^2}\right)\right]\right|\right|_{L^\infty} \\ &\leqslant Ce^{-\delta t} + Ct^{-1}e^{-\delta t} + Ct^2e^{-\frac{t}{2}}\left|\left|\mathcal{F}^{-1}\left[\chi_h(\xi)\left(\frac{\cos(t|\xi|)}{|\xi|^2}\right)\right]\right|\right|_{L^\infty} \end{aligned} \tag{4.49}$$

for $0<\delta<\frac{1}{2}$. The last term of (4.49) can be treated by the stationary phase method. By Lemma 4.3, we conclude that

$$\begin{aligned} ||\mathcal{F}^{-1}[m_h]||_{L^\infty} &\leqslant Ct^2e^{-\frac{t}{2}}\left|\left|\mathcal{F}^{-1}\left[\chi_h(\xi)\left(\frac{\cos(t|\xi|)}{|\xi|^2}\right)\right]\right|\right|_{L^\infty} + Ct^{-1}e^{-\delta t} \\ &\leqslant Ct^{-1}e^{-\delta t}. \end{aligned} \tag{4.50}$$

This proves (4.20).

Finally, we prove (4.21). Again by the Hausdorff–Young inequality, it suffices to show that

$$||\mathcal{F}^{-1}[m_l(\xi)]||_{L^\infty} \leqslant C(1+t)^{-2}.$$

Putting $\beta(\xi) = \sqrt{\frac{1}{4} - |\xi|^2}$, we estimate by dividing the multiplier as

$$\begin{aligned} &||\mathcal{F}^{-1}[m_l(\xi)]||_{L^\infty} \\ &\leqslant e^{-\frac{t}{2}} \left\| \chi_l(\xi)\left(\cos(t|\xi|) - \frac{t}{8}\frac{\sin(t|\xi|)}{|\xi|}\right)\right\|_{L^1} + \left\|\chi_l(\xi)\left(e^{-\frac{t}{2}}\cosh\{t\beta(\xi)\} - \frac{1}{2}e^{-t|\xi|^2}\right)\right\|_{L^1} \\ &\leqslant Ce^{-\frac{t}{2}} + \left\|\chi_l(\xi)\left(e^{-\frac{t}{2}}\cosh\{t\beta(\xi)\} - \frac{1}{2}e^{-t|\xi|^2}\right)\right\|_{L^1}. \end{aligned} \tag{4.51}$$

From (4.30), it follows

$$\begin{aligned} &\left\|\chi_l(\xi)\left(e^{-\frac{t}{2}}\cosh\{t\beta(\xi)\} - \frac{1}{2}e^{-t|\xi|^2}\right)\right\|_{L^1} \\ &\leqslant C\int_{|\xi|\leqslant\frac{1}{4}} t|\xi|^4 e^{-t|\xi|^2}\, d\xi + Ce^{-t}\int_{|\xi|\leqslant\frac{1}{4}} e^{-\frac{t}{2}(-1+\sqrt{1-4|\xi|^2})}d\xi \\ &\leqslant C(1+t)^{-2} + Ce^{-\delta t} \end{aligned} \tag{4.52}$$

for $0<\delta<1/2$. Thus by (4.51),

$$\begin{aligned} ||\mathcal{F}^{-1}[m_l(\xi)]||_{L^\infty} &\leqslant Ce^{-\frac{t}{2}} + C(1+t)^{-2} + Ce^{-\delta t} \\ &\leqslant C(1+t)^{-2}. \end{aligned} \tag{4.53}$$

This concludes (4.21) and Proposition 4.2 has proved. □

Remark. Let $g\in L^q(\mathbb{R}^2)$. Taking a similar way to the above, one has

$$||(K_1(t) - e^{-\frac{t}{2}}W_1(t))g||_{L^p} \leqslant C(1+t)^{-\gamma}||g||_{L^q} \tag{4.54}$$

for all $t\in\mathbb{R}_+$ and $\gamma = \frac{1}{q} - \frac{1}{p}$. Note that the bound $(1+t)^{-\gamma}$ in (4.54) is bounded function for small t. It is well known that the solution v of (2.2) satisfies

$$||v(t)||_{L^p} \leqslant C(||u_0||_{L^p} + ||u_1||_{L^p}).$$

Making use of (4.54) with $p=q$, we have

$$||(K_1(t) - e^{-\frac{t}{2}}W_1(t))(u_0+u_1) - v(t)||_{L^p} \leqslant C(||u_0||_{L^p} + ||u_1||_{L^p})$$

for $0 \leqslant t \leqslant 1$. Thus, adding to (4.7) with $t \geqslant 1$, the estimate

$$||(K_1(t) - e^{-\frac{t}{2}} W_1(t))(u_0 + u_1) - v(t)||_{L^p} \leqslant C(1+t)^{-1}(||u_0||_{L^p} + ||u_1||_{L^p}) \tag{4.55}$$

holds for $1 \leqslant p \leqslant \infty$. Both (4.54) and (4.55) is necessary to prove Theorems 2.2 and 2.3.

5. Proof of Theorem 2.2

First we give a linear estimate for the homogeneous part (2.1).

Lemma 5.1 (Matsumura [21]). *Let $u_0 \in B^1_{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and $u_1 \in B^0_{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Then the solution $u(t,x) = K(u_0, u_1)$ to* (2.1) *satisfies the following estimates*:

$$||K(u_0, u_1)||_{L^2} \leqslant C(1+t)^{-\frac{1}{2}}(||u_0||_{L^2} + ||u_0||_{L^1} + ||u_1||_{L^2} + ||u_1||_{L^1}), \tag{5.1}$$

$$||K(u_0, u_1)||_{L^\infty} \leqslant C(1+t)^{-1}(||u_0||_{B^1_{2,1}} + ||u_0||_{L^1} + ||u_1||_{B^0_{2,1}} + ||u_1||_{L^1}), \tag{5.2}$$

$$||\nabla K(u_0, u_1)||_{L^2} \leqslant C(1+t)^{-1}(||u_0||_{B^1_{2,1}} + ||u_0||_{L^1} + ||u_1||_{B^0_{2,1}} + ||u_1||_{L^1}). \tag{5.3}$$

Proof. The first estimate (5.1) is shown by Matsumura [21]. The last estimate (5.3) follows from the estimate

$$||\nabla K(u_0, u_1)||_{L^2} \leqslant C(1+t)^{-1}(||u_0||_{H^1} + ||u_0||_{L^1} + ||u_1||_{L^2} + ||u_1||_{L^1}),$$

which is also obtained by Matsumura [21]. We show (5.2). By (3.3), it suffices to prove that

$$||K_0(t)u_0||_{L^\infty} \leqslant C(1+t)^{-1}(||u_0||_{B^1_{2,1}} + ||u_0||_{L^1}), \tag{5.4}$$

$$||K_1(t)u_1||_{L^\infty} \leqslant C(1+t)^{-1}(||u_1||_{B^0_{2,1}} + ||u_1||_{L^1}). \tag{5.5}$$

We only show the first inequality (5.4). The other case can be proved by a similar way. By Hausdorff–Young's inequality,

$$\begin{aligned}
||K_0(t)u_0||_{L^\infty} &\leqslant C||\hat{K}_0(t)\hat{u}_0||_{L^1} \\
&= C\Bigg(\int_{|\xi|<\frac{1}{2}} e^{-\frac{t}{2}}|\cos\{t\alpha(\xi)\}\hat{u}_0(\xi)|\,d\xi + \int_{\frac{1}{2}\leqslant|\xi|\leqslant 2} e^{-\frac{t}{2}}|\cos\{t\alpha(\xi)\}\hat{u}_0(\xi)|\,d\xi \\
&\quad + \int_{|\xi|>2} e^{-\frac{t}{2}}|\cos\{t\alpha(\xi)\}\hat{u}_0(\xi)|\,d\xi\Bigg) \\
&=: C(I_1+I_2+I_3).
\end{aligned} \tag{5.6}$$

Using (4.30),

$$\begin{aligned}
I_1 &= \int_{|\xi|<\frac{1}{2}} e^{-\frac{t}{2}}|\cos\{t\alpha(\xi)\}\hat{u}_0(\xi)|\,d\xi \\
&\leqslant C||\hat{u}_0||_{L^\infty}\int_{|\xi|<\frac{1}{2}}\left|e^{-\frac{t}{2}(1-\sqrt{1-4|\xi|^2})}+e^{-\frac{t}{2}(1+\sqrt{1-4|\xi|^2})}\right|d\xi \\
&\leqslant C(1+t)^{-1}||u_0||_{L^1}.
\end{aligned} \tag{5.7}$$

The estimate of I_2 is straightforward.

$$\begin{aligned}
I_2 &= \int_{\frac{1}{2}\leqslant|\xi|\leqslant 2} e^{-\frac{t}{2}}|\cos\{t\alpha(\xi)\}\hat{u}_0(\xi)|\,d\xi \\
&\leqslant C||\hat{u}_0||_{L^\infty}e^{-\frac{t}{2}}\int_{\frac{1}{2}\leqslant|\xi|\leqslant 2}d\xi \\
&\leqslant Ce^{-\frac{t}{2}}||u_0||_{L^1}.
\end{aligned} \tag{5.8}$$

To estimate I_3, we recall $\{\hat{\phi}_j(\xi)\}_{j=-\infty}^{\infty}$ be the Littlewood–Paley dyadic decomposition. Let

$$\tilde{\hat{\phi}}_j(\xi) := \hat{\phi}_{j-1}(\xi)+\hat{\phi}_j(\xi)+\hat{\phi}_{j+1}(\xi).$$

Then

$$\begin{aligned}
I_3 &= \int_{|\xi|>2} e^{-\frac{t}{2}}|\cos\{t\alpha(\xi)\}\hat{u}_0(\xi)|\,d\xi \\
&\leqslant Ce^{-\frac{t}{2}}\sum_{j=1}^{\infty}||\cos\{t\alpha(\xi)\}\tilde{\hat{\phi}}_j(\xi)\hat{\phi}_j\hat{u}_0||_{L^1} \\
&\leqslant Ce^{-\frac{t}{2}}\sum_{j=1}^{\infty}||\tilde{\hat{\phi}}_j(\xi)||_{L^2}||\hat{\phi}_j\hat{u}_0||_{L^2}
\end{aligned}$$

$$\leqslant Ce^{-\frac{t}{2}} \sum_{j=1}^{\infty} 2^j ||\phi_j * u_0||_{L^2}$$

$$\leqslant Ce^{-\frac{t}{2}} ||u_0||_{B^1_{2,1}}. \tag{5.9}$$

By (5.6)–(5.9), we have (5.4). $\square$

Lemma 5.2 (Matsumura [21]). *Let K_1 be the operator defined in* (3.2). *For $f \in H^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, we have*

$$||K_1 f||_{L^2} \leqslant C||f||_{L^2}, \tag{5.10}$$

$$||\nabla K_1 f||_{L^2} \leqslant C(||f||_{L^1} + ||f||_{L^2}). \tag{5.11}$$

See [21] for the proof of Lemma 5.2.

To show the existence of solution, we introduce the successive approximation sequence $\{u_{(n)}\}_{n=0}^{\infty}$ defined as follows:

$$\begin{cases} u_{(0)}(t,x) = K(u_0, u_1), \\ u_{(n+1)}(t,x) = K(u_0, u_1) + \int_0^t K_1(t-s)|u_{(n)}|^{\alpha} u_{(n)}(s, \cdot)\, ds. \end{cases} \tag{5.12}$$

We also define the metric space X, Y as follows.

$$X \coloneqq \{(u,v) \in (B^1_{2,1} \cap L^1) \times (B^0_{2,1} \cap L^1) \,:\, ||u,v||_X < \infty\}, \tag{5.13}$$

$$Y \coloneqq \{u \in C([0,\infty); L^2 \cap L^\infty) \,:\, ||u||_Y < M\}, \tag{5.14}$$

where

$$||u,v||_X \coloneqq ||u||_{B^1_{2,1}} + ||u||_{L^1} + ||v||_{B^0_{2,1}} + ||v||_{L^1}, \tag{5.15}$$

$$||u||_Y \coloneqq \sup_{t \in [0,\infty)} \{(1+t)^{\frac{1}{2}} ||u||_{L^2} + (1+t)||u||_{L^\infty}\}. \tag{5.16}$$

Proposition 5.3. *Let $\{u_{(n)}\}_{n \in \mathbb{N}}$ be the sequence defined by* (5.12). *If $||u_0, u_1||_X$ is sufficiently small, then there exists a constant $C > 0$ such that*

$$||u_{(n)}||_Y \leqslant C||u_0, u_1||_X \tag{5.17}$$

for all $n \in \mathbb{N}$.

Proof. We prove (5.17) by the induction argument. From (5.1) and (5.2),

$$||u_{(0)}(t)||_{L^2} \leqslant C(1+t)^{-\frac{1}{2}}(||u_0||_{B^1_{2,1}} + ||u_0||_{L^1} + ||u_1||_{B^0_{2,1}} + ||u_1||_{L^1}),$$
$$||u_{(0)}(t)||_{L^\infty} \leqslant C(1+t)^{-1}(||u_0||_{B^1_{2,1}} + ||u_0||_{L^1} + ||u_1||_{B^0_{2,1}} + ||u_1||_{L^1})$$

and we see

$$\begin{aligned}||u_{(0)}||_Y &= \sup_{t\in[0,\infty)}\{(1+t)^{\frac{1}{2}}||u_{(0)}(t)||_{L^2} + (1+t)||u_{(0)}(t)||_{L^\infty}\}\\ &\leqslant C_0||u_0,u_1||_X. \qquad (5.18)\end{aligned}$$

Next we assume that $||u_{(n)}||_Y \leqslant 2C_0||u_0,u_1||_X$ for $n \geqslant 1$. Since $\alpha > 1$, we have from (5.10) and (5.12) that

$$\begin{aligned}||u_{(n+1)}||_{L^2} &\leqslant ||u_{(0)}||_{L^2} + C\int_0^t ||K_1(t-s)|u_{(n)}|^\alpha u_{(n)}(s)||_{L^2}\,ds\\ &\leqslant ||u_{(0)}||_{L^2} + C||u_{(n)}||_X^{\alpha+1}\int_0^t (1+t-s)^{-\frac{1}{2}}(1+s)^{-\alpha}\,ds\\ &\leqslant ||u_{(0)}||_{L^2} + C_1(1+t)^{-\frac{1}{2}}||u_{(n)}||_X^{\alpha+1}. \qquad (5.19)\end{aligned}$$

Using (3.9) in Lemma 3.4 and (4.54) with $p=\infty$, $q=1$, it follows

$$\begin{aligned}||u_{(n+1)}||_{L^\infty} &\leqslant ||u_{(0)}||_{L^\infty} + C\int_0^t ||(K_1(t-s) - e^{-\frac{t-s}{2}}W_1(t-s))|u_{(n)}|^\alpha u_{(n)}(s)||_{L^\infty}\,ds\\ &\quad + C\int_0^t e^{-\frac{t-s}{2}}||W_1(t-s)|u_{(n)}|^\alpha u_{(n)}(s)||_{L^\infty}\,ds\\ &\leqslant ||u_{(0)}||_{L^\infty} + C\int_0^t (1+t-s)^{-1}|||u_{(n)}|^{\alpha+1}||_{L^1}\,ds\\ &\quad + C\int_0^t e^{-\frac{t-s}{2}}(t-s)|||u_{(n)}|^{\alpha+1}||_{L^\infty}\,ds\\ &\leqslant ||u_{(0)}||_{L^\infty} + C\int_0^t (1+t-s)^{-1}(1+s)^{-\alpha}\,ds||u_{(n)}||_X^{\alpha+1}\\ &\quad + C\int_0^t e^{-\frac{t-s}{2}}(t-s)(1+s)^{-\alpha-1}\,ds||u_{(n)}||_X^{\alpha+1}\\ &\leqslant ||u_{(0)}||_{L^\infty} + C_2(1+t)^{-1}||u_{(n)}||_X^{\alpha+1}. \qquad (5.20)\end{aligned}$$

Hence (5.19) and (5.20) yield

$$||u_{(n+1)}||_Y \leqslant ||u_{(0)}||_Y + C||u_{(n)}||_Y^{\alpha+1},$$

where C is $\max(C_1, C_2)$. Therefore if $2C(2C_0||u_0, u_1||_X)^\alpha < 1$, then we have $||u_{(n+1)}||_Y \leqslant 2C_0||u_0, u_1||_X$. $\square$

To see the approximation sequence $\{u_{(n)}\}_{n\in\mathbb{N}}$ is Cauchy, we need the following proposition.

Proposition 5.4. *Let* $\{u_{(n)}\}_{n\in\mathbb{N}}$ *be the sequence defined by* (5.12). *Then it holds that for all* $n\in\mathbb{N}$,

$$||u_{(n+1)} - u_{(n)}||_Y \leqslant \frac{1}{2}||u_{(n)} - u_{(n-1)}||_Y. \tag{5.21}$$

Proof. By definition (5.12),

$$\begin{aligned}
&(u_{(n+1)} - u_{(n)})(t)\\
&\quad = \int_0^t K_1(t-s)(|u_{(n)}|^\alpha u_{(n)}(s) - |u_{(n-1)}|^\alpha u_{(n-1)}(s))\,ds\\
&\quad = \int_0^t (K_1(t-s) - e^{-\frac{t-s}{2}}W_1(t-s))(|u_{(n)}|^\alpha u_{(n)}(s) - |u_{(n-1)}|^\alpha u_{(n-1)}(s))\,ds\\
&\qquad + \int_0^t e^{-\frac{t-s}{2}}W_1(t-s)(|u_{(n)}|^\alpha u_{(n)}(s) - |u_{(n-1)}|^\alpha u_{(n-1)}(s))\,ds.
\end{aligned} \tag{5.22}$$

Noting that

$$|a|^\alpha a - |b|^\alpha b = \int_0^t (\alpha+1)|\theta a + (1-\theta)b|^\alpha (a-b)\,d\theta$$

for any $a, b > 0$, it follows by Minkowski's inequality that

$$\begin{aligned}
&||(|u_{(n)}|^\alpha u_{(n)}(s) - |u_{(n-1)}|^\alpha u_{(n-1)}(s))||_{L^2}\\
&\quad \leqslant C(|||u_{(n)}|^\alpha||_{L^\infty} + |||u_{(n-1)}|^\alpha||_{L^\infty})||u_{(n)} - u_{(n-1)}||_{L^2}\\
&\quad \leqslant C(||u_{(n)}||_{L^\infty}^\alpha + ||u_{(n-1)}||_{L^\infty}^\alpha)||u_{(n)} - u_{(n-1)}||_{L^2}.
\end{aligned}$$

Since the evolution operator $K_1(t)$ is L^2-bounded by (5.11) in Proposition 5.2,

$$\begin{aligned}
&||(u_{(n+1)} - u_{(n)})(t)||_{L^2} \\
&\leqslant C\int_0^t ||K_1(t-s)(|u_{(n)}|^\alpha u_{(n)}(s) - |u_{(n-1)}|^\alpha u_{(n-1)}(s))||_{L^2}\, ds \\
&\leqslant C\int_0^t (||u_{(n)}||^\alpha_{L^\infty} + ||u_{(n-1)}||^\alpha_{L^\infty})||u_{(n)} - u_{(n-1)}||_{L^2}\, ds \\
&\leqslant C(||u_{(n)}||^\alpha_Y + ||u_{(n-1)}||^\alpha_Y)||u_{(n)} - u_{(n-1)}||_Y \int_0^t (1+s)^{-\frac{1}{2}-\alpha}\, ds \\
&\leqslant C_3(1+t)^{-\frac{1}{2}}||u_0,u_1||^\alpha_X ||u_{(n)} - u_{(n-1)}||_Y.
\end{aligned} \tag{5.23}$$

Analogously we use (3.9) in Lemma 3.4 and (4.54) with $p=\infty$, $q=1$ to obtain

$$\begin{aligned}
&||(u_{(n+1)} - u_{(n)})(t)||_{L^\infty} \\
&\leqslant C\int_0^t ||(K_1(t-s) - e^{-\frac{t-s}{2}}W_1(t-s))(|u_{(n)}|^\alpha u_{(n)}(s) - |u_{(n-1)}|^\alpha u_{(n-1)}(s))||_{L^\infty}\, ds \\
&\quad + C\int_0^t e^{-\frac{t-s}{2}}||W_1(t-s)(|u_{(n)}|^\alpha u_{(n)}(s) - |u_{(n-1)}|^\alpha u_{(n-1)}(s))||_{L^\infty}\, ds \\
&\leqslant C\int_0^t (1+t-s)^{-1}||(|u_{(n)}|^\alpha u_{(n)}(s) - |u_{(n-1)}|^\alpha u_{(n-1)}(s))||_{L^1}\, ds \\
&\quad + C\int_0^t e^{-\frac{t-s}{2}}(t-s)||(|u_{(n)}|^\alpha u_{(n)}(s) - |u_{(n-1)}|^\alpha u_{(n-1)}(s))||_{L^\infty}\, ds \\
&\leqslant C\int_0^t (1+t-s)^{-1}(1+s)^{-\alpha}\, ds(||u_{(n)}||^\alpha_Y + ||u_{(n-1)}||^\alpha_Y)||u_{(n)} - u_{(n-1)}||_Y \\
&\quad + C\int_0^t e^{-\frac{t-s}{2}}(t-s)(1-s)^{-\alpha-1}\, ds(||u_{(n)}||^\alpha_Y + ||u_{(n-1)}||^\alpha_Y)||u_{(n)} - u_{(n-1)}||_Y \\
&\leqslant C_4(1+t)^{-1}||u_0,u_1||^\alpha_X ||u_{(n)} - u_{(n-1)}||_Y.
\end{aligned} \tag{5.24}$$

From (5.23) and (5.24), we conclude

$$||u_{(n+1)} - u_{(n)}||_Y \leqslant \frac{1}{2}||u_{(n)} - u_{(n-1)}||_Y \tag{5.25}$$

under the condition $\max(C_3, C_4)||u_0,u_1||^\alpha_X \leqslant \frac{1}{2}$. Hence we proved the proposition. □

Proof of Theorem 2.2. We fix a constant C_0 so that (5.18) is satisfied. Put $M = 2C_0||u_0,u_1||_X$. Since by Propositions 5.3 and 5.4, the sequence $\{u_{(n)}\}_{n\in\mathbb{N}}$, defined by (5.12), is Cauchy in Y. Therefore since Y is complete metric space,

$$u := \lim_{n\to\infty} u_{(n)}$$

uniquely exists in Y. By the fixed point theorem, we have the solution u to the integral equation (3.4) and u satisfies

$$\begin{aligned} &u \in C([0,\infty); L^2 \cap L^\infty), \\ &\|u(t)\|_{L^2} \leqslant C(1+t)^{-\frac{1}{2}}, \\ &\|u(t)\|_{L^\infty} \leqslant C(1+t)^{-1}, \end{aligned} \tag{5.26}$$

namely we obtain (2.8). Moreover, using (5.3) and (5.11),

$$\begin{aligned} \|\nabla u(t)\|_{L^2} &\leqslant \|\nabla K(u_0,u_1)\|_{L_2} + \int_0^t \|\nabla K_1(t-s)|u|^\alpha u(s)\|_{L^2}\, ds \\ &\leqslant C(1+t)^{-1}\|u_0,u_1\|_X + \int_0^t (1+t-s)^{-1}(\||u|^\alpha u(s)\|_{L^1} + \||u|^\alpha u(s)\|_{L^2})\, ds \\ &\leqslant C(1+t)^{-1}\|u_0,u_1\|_X + \int_0^t (1+t-s)^{-1}(1+s)^{-\alpha}\, ds\|u_0,u_1\|_X \\ &\leqslant C(1+t)^{-1}\|u_0,u_1\|_X. \end{aligned} \tag{5.27}$$

Therefore, we have (2.9) and (2.10). This completes the proof of Theorem 2.2. □

6. Proof of Theorem 2.3

In Theorem 2.3, we show that the semilinear damped wave equation can be also approximated by the solutions of heat and wave equations as well as the linear equation. We prove Theorem 2.3 by applying the linear estimates and the decay estimate obtained in Theorems 2.1 and 2.2.

Proof of Theorem 2.3. Let u, $\bar{v}$, w, $\tilde{w}$ be the solutions to (1.1), (2.11), (2.3), (2.4), respectively. Then using the solution v of (2.2), we have

$$\begin{aligned} &u(t) - \bar{v}(t) - e^{-\frac{t}{2}}\left\{w(t) + \left(\frac{1}{2}+\frac{t}{8}\right)\tilde{w}(t)\right\} \\ &\quad = u(t) - v(t) - e^{-\frac{t}{2}}\left\{w(t) + \left(\frac{1}{2}+\frac{t}{8}\right)\tilde{w}(t)\right\} \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t e^{-\frac{t-s}{2}} W_1(t-s)|u|^\alpha u(s)\,ds \\
&+\int_0^t \{(K_1(t-s)-e^{-\frac{t-s}{2}} W_1(t-s))-e^{(t-s)\Delta}\}|u|^\alpha u(s)\,ds.
\end{aligned} \tag{6.1}
$$

By Theorem 2.1,

$$
\begin{aligned}
&\left\|u(t)-\bar{v}(t)-e^{-\frac{t}{2}}\left\{\bar{w}(t)+\left(\frac{1}{2}+\frac{t}{8}\right)\tilde{w}(t)\right\}\right\|_{L^p} \\
&\quad = Ct^{-(1-\frac{1}{p})-1}(||u_0||_{L^1}+||u_1||_{L^1}) \\
&\qquad +\int_0^t e^{-\frac{t-s}{2}}||W_1(t-s)|u|^\alpha u(s)||_{L^p}\,ds \\
&\qquad +\int_0^t ||\{(K_1(t-s)-e^{-\frac{t-s}{2}} W_1(t-s))-e^{(t-s)\Delta}\}|u|^\alpha u(s)||_{L^p}\,ds.
\end{aligned} \tag{6.2}
$$

Using (3.8), the second term on the right-hand side of (6.2) is estimated as follow:

$$
\begin{aligned}
&\int_0^t e^{-\frac{t-s}{2}}||W_1(t-s)|u|^\alpha u(s)||_{L^p}\,ds \\
&\quad \leqslant \int_0^t e^{-\frac{t-s}{2}}(t-s)^{\frac{4}{p}}|||u|^\alpha \nabla u(s)||_{L^{p'}}\,ds \\
&\quad \leqslant \int_0^t e^{-\frac{t-s}{2}}(t-s)^{\frac{4}{p}}||u||^\alpha_{\frac{2\alpha p'}{2-p'}}||\nabla u(s)||_{L^2}\,ds \\
&\quad \leqslant \int_0^t e^{-\frac{t-s}{2}}(t-s)^{\frac{4}{p}}(1+s)^{-(\alpha+\frac{1}{p}+\frac{1}{2})}\,ds||u_0,u_1||_X.
\end{aligned} \tag{6.3}
$$

Thus, if $\alpha > \frac{1}{2}-\frac{1}{p}$, then

$$
\int_0^t e^{-\frac{t-s}{2}}||W_1(t-s)|u|^\alpha u(s)||_{L^p}\,ds \leqslant C(1+t)^{-(1-\frac{1}{p})-\varepsilon}||u_0,u_1||_X. \tag{6.4}
$$

To estimate the third term of (6.2), we use the identity; for $a,b,c>0$,

$$
a^2-b^2-c^2=(a-b)(a-b-c)+(a-b-c)c+2(a-b)b.
$$

Observing that all of the evolution operators $K_1(t)$, $e^{-t/2}W_1(t)$ and $e^{t\Delta}$ commute each other, it follows

$$\begin{aligned}
&\int_0^t ||\{(K_1(t-s) - e^{-\frac{t-s}{2}}W_1(t-s)) - e^{(t-s)\Delta}\}|u|^\alpha u(s)||_{L^p}\, ds \\
&\leqslant \int_0^t \bigg|\bigg|\Big\{K_1\Big(\frac{t-s}{2}\Big) - e^{-\frac{t-s}{4}}W_1\Big(\frac{t-s}{2}\Big)\Big\} \\
&\quad \times \Big\{K_1\Big(\frac{t-s}{2}\Big) - e^{-\frac{t-s}{4}}W_1\Big(\frac{t-s}{2}\Big) - e^{(\frac{t-s}{2})\Delta}\Big\}|u|^\alpha u(s)\bigg|\bigg|_{L^p} ds \\
&\quad + \int_0^t \bigg|\bigg|\Big\{K_1\Big(\frac{t-s}{2}\Big) - e^{-\frac{t-s}{4}}W_1\Big(\frac{t-s}{2}\Big) - e^{(\frac{t-s}{2})\Delta}\Big\}e^{(\frac{t-s}{2})\Delta}|u|^\alpha u(s)\bigg|\bigg|_{L^p} ds \\
&\quad + 2\int_0^t \bigg|\bigg|e^{-\frac{t-s}{4}}\Big\{K_1\Big(\frac{t-s}{2}\Big) - e^{-\frac{t-s}{4}}W_1\Big(\frac{t-s}{2}\Big)\Big\}W_1\Big(\frac{t-s}{2}\Big)|u|^\alpha u(s)\bigg|\bigg|_{L^p} ds \\
&=: I_1 + I_2 + 2I_3. \qquad (6.5)
\end{aligned}$$

By (4.54) in the remark after proof of Theorem 2.1, the first term I_1 is

$$\begin{aligned}
I_1 &\leqslant C\int_0^t (1+t-s)^{-1}\bigg|\bigg|\Big\{K_1\Big(\frac{t-s}{2}\Big) - e^{-\frac{t-s}{4}}W_1\Big(\frac{t-s}{2}\Big)\Big\}|u|^\alpha u(s)\bigg|\bigg|_{L^p} ds \\
&\leqslant C\int_0^t (1+t-s)^{-1}(1+t-s)^{-(1-\frac{1}{p})}|||u|^\alpha u(s)||_{L^1}\, ds \\
&\leqslant C\int_0^t (1+t-s)^{-(1-\frac{1}{p})-1}(1+s)^{-\alpha}\, ds||u_0,u_1||_X.
\end{aligned}$$

Therefore, if $\alpha > 1$, Lemma 3.3 gives

$$I_1 \leqslant C(1+t)^{-(1-\frac{1}{p})-\varepsilon}||u_0,u_1||_X. \qquad (6.6)$$

By (4.55) in the remark after proof of Theorem 2.1 and the $L^p - L^q$ estimate of heat equation, the estimate of the second term I_2 follows

$$\begin{aligned}
I_2 &\leqslant C\int_0^t (1+t-s)^{-1}||e^{(\frac{t-s}{2})\Delta}|u|^\alpha u(s)||_{L^p}\, ds \\
&\leqslant C\int_0^{\frac{t}{2}} (1+t-s)^{-1}(t-s)^{-(1-\frac{1}{p})}(1+s)^{-\alpha}\, ds||u_0,u_1||_X \\
&\quad + C\int_{\frac{t}{2}}^t (1+t-s)^{-1}(t-s)^{-(\frac{1}{2}-\frac{1}{p})}(1+s)^{-\alpha}\, ds||u_0,u_1||_X
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C(1+t)^{-1}t^{-(1-\frac{1}{p})}\int_0^{\frac{t}{2}}(1+s)^{-\alpha}\,ds||u_0,u_1||_X\\
&\quad + C\int_0^{\frac{t}{2}}(1+r)^{-1}r^{-(\frac{1}{2}-\frac{1}{p})}(1+t-r)^{-\alpha}\,dr||u_0,u_1||_X\\
&\leqslant C(1+t)^{-1}t^{-(1-\frac{1}{p})}||u_0,u_1||_X + C\int_0^1(1+r)^{-1}r^{-(\frac{1}{2}-\frac{1}{p})}(1+t-r)^{-\alpha}\,dr||u_0,u_1||_X\\
&\quad + C\int_1^{\frac{t}{2}}(1+r)^{-1}r^{-(\frac{1}{2}-\frac{1}{p})}(1+t-r)^{-\alpha}\,dr||u_0,u_1||_X\\
&\leqslant C(1+t)^{-1}t^{-(1-\frac{1}{p})}||u_0,u_1||_X + Ct^{-\alpha}\int_0^1 r^{-(\frac{1}{2}-\frac{1}{p})}\,dr||u_0,u_1||_X\\
&\quad + C(1+t)^{-\alpha}\int_1^{\frac{t}{2}} r^{-(\frac{3}{2}-\frac{1}{p})}\,dr||u_0,u_1||_X.
\end{aligned}
$$

Hence, if $\alpha > 1$, then

$$
I_2 \leqslant Ct^{-(1-\frac{1}{p})-1}||u_0,u_1||_X + C(1+t)^{-(1-\frac{1}{p})-\varepsilon}||u_0,u_1||_X. \tag{6.7}
$$

Using (4.54) and the $L^p - L^q$ estimate of wave equation, it follows that

$$
\begin{aligned}
I_3 &= \int_0^t \left|\left| e^{-\frac{t-s}{4}}\left\{K_1\left(\frac{t-s}{2}\right) - e^{-\frac{t-s}{4}}W_1\left(\frac{t-s}{2}\right)\right\}W_1\left(\frac{t-s}{2}\right)|u|^\alpha u(s)\right|\right|_{L^p} ds\\
&\leqslant \int_0^t e^{-\frac{t-s}{4}}(1+t-s)^{-(\frac{1}{2}-\frac{1}{p})}\left|\left|W_1\left(\frac{t-s}{2}\right)|u|^\alpha u(s)\right|\right|_{L^2} ds\\
&\leqslant \int_0^t e^{-\frac{t-s}{4}}(1+t-s)^{-(\frac{1}{2}-\frac{1}{p})}(t-s)|||u|^\alpha u(s,\cdot)||_{L^2}\,ds\\
&\leqslant \int_0^t e^{-\frac{t-s}{4}}(t-s)(1+s)^{-\alpha}\,ds||u_0,u_1||_X.
\end{aligned} \tag{6.8}
$$

Thus, if $\alpha > 1$, Lemma 3.3 implies

$$
I_3 \leqslant (1+s)^{-(1-\frac{1}{p})-\varepsilon}||u_0,u_1||_X. \tag{6.9}
$$

Making use of (6.2), (6.4), (6.5), (6.7) and (6.9),

$$\left\| u(t) - \bar{v}(t) - e^{-\frac{t}{2}} \left\{ w(t) + \left(\frac{1}{2} + \frac{t}{8} \right) \tilde{w}(t) \right\} \right\|_{L^p} \leqslant C t^{-(1-\frac{1}{p})-\varepsilon} \| u_0, u_1 \|_X$$

for $2 \leqslant p \leqslant \infty$, $\alpha > 1$. Thus we have completed the proof of Theorem 2.3. □

In view of Theorem 2.3, the proof for the Corollary 2.4 only requires to show the asymptotic profile of the solution of $\bar{v}(t)$. The following proposition directly yields the result.

Proposition 6.1. *Let u be a solution of the semi-linear damped wave equation and $\bar{v}(t)$ solves the homogeneous heat equation* (2.11). *Then we have*

$$\left\| \bar{v}(t) - (M_0 + M_1) G_t - G_t \int_0^\infty \int_{\mathbb{R}^2} |u(s)|^\alpha u(s)\, dx\, ds \right\|_{L^p} = o\left(t^{-(1-1/p)} \right), \qquad (6.10)$$

where $M_i = \int_{\mathbb{R}^2} u_i(x)\, dx$ is the average of the initial data and $G_t = G_t(x) = \frac{1}{4\pi t} e^{-|x|^2/4t}$ is the heat kernel.

Proof. Since from Theorem 2.2

$$\| u(s) \|_{L^{\alpha+1}}^{\alpha+1} \leqslant C(1+s)^{-\alpha}$$

and $\alpha > 2/n = 1$ implies that the above norm is integrable over $t \in [0, \infty)$.

To show the claim it is enough to show that

$$t^{1-1/p} \left\| \int_0^t e^{(t-s)\Delta} |u(s)|^{\alpha-1} u(s)\, ds - G_t \int_0^\infty \int_{\mathbb{R}^2} |u(s)|^\alpha u(s)\, dx\, ds \right\|_{L^p} \to 0, \qquad (6.11)$$

since the term from the initial data is shown much easier way.

Observing the change of variable as

$$
\begin{aligned}
&t^{1-1/p}\left\|\int_0^t e^{(t-s)\Delta}|u(s)|^\alpha u(s)\,ds - G_t\int_0^\infty\int_{\mathbb{R}^2}|u(s)|^\alpha u(s)\,dx\,ds\right\|_{L^p(\mathbb{R}^2_x)}\\
&\quad = t^{-1/p}\left\|\int_0^t\int_{\mathbb{R}^2} G_{(1-t^{-1}s)}\left(\frac{x-y}{\sqrt{t}}\right)|u(s,y)|^\alpha u(s,y)\,dy\,ds\right.\\
&\qquad - G_1\left(\frac{x}{\sqrt{t}}\right)\int_0^t\int_{\mathbb{R}^2}|u(s,y)|^\alpha u(s,y)\,dy\,ds\\
&\qquad \left. - G_1\left(\frac{x}{\sqrt{t}}\right)\int_t^\infty\int_{\mathbb{R}^2}|u(s,y)|^\alpha u(s,y)\,dy\,ds\right\|_{L^p(\mathbb{R}^2_x)}\\
&\quad \leqslant \left\|\int_0^t\int_{\mathbb{R}^2} G_{(1-t^{-1}s)}\left(x'-\frac{y}{\sqrt{t}}\right)|u(s,y)|^\alpha u(s,y)\,dy\,ds\right.\\
&\qquad \left. - G_1(x')\int_0^t\int_{\mathbb{R}^2}|u(s,y)|^\alpha u(s,y)\,dy\,ds\right\|_{L^p(\mathbb{R}^2_{x'})} + C(1+t)^{-\alpha+1}. \qquad (6.12)
\end{aligned}
$$

Since $\alpha > 1$, the last term decay faster as $t \to \infty$. Hence the main term is the first term. Let us exchange x' into x for simplicity. For small $\delta > 0$,

$$
\begin{aligned}
&\left\|\int_0^t\int_{\mathbb{R}^2}\left(G_{(1-t^{-1}s)}\left(x-\frac{y}{\sqrt{t}}\right)-G_1(x)\right)|u(s,y)|^\alpha u(s,y)\,dy\,ds\right\|_{L^p}\\
&\quad \leqslant \left\|\int_0^{\delta t}\int_{B_{\delta\sqrt{t}}}\left(G_{(1-t^{-1}s)}\left(x-\frac{y}{\sqrt{t}}\right)-G_1(x)\right)|u(s,y)|^\alpha u(s,y)\,dy\,ds\right\|_{L^p}\\
&\qquad + \left\|\int_0^{\delta t}\int_{B^c_{\delta\sqrt{t}}}\left(G_{(1-t^{-1}s)}\left(x-\frac{y}{\sqrt{t}}\right)-G_1(x)\right)|u(s,y)|^\alpha u(s,y)\,dy\,ds\right\|_{L^p}\\
&\qquad + \left\|\int_{\delta t}^t\int_{\mathbb{R}^2}\left(G_{(1-t^{-1}s)}\left(x-\frac{y}{\sqrt{t}}\right)-G_1(x)\right)|u(s,y)|^\alpha u(s,y)\,dy\,ds\right\|_{L^p}\\
&\quad \equiv I_1 + I_2 + I_3. \qquad (6.13)
\end{aligned}
$$

The estimate for I_1. For $y \in B_{\delta\sqrt{t}}$, we can choose δ sufficiently small that

$$
\begin{aligned}
&\left|G_{1-t^{-1}s}\left(x-\frac{y}{\sqrt{t}}\right)-G_1(x)\right|\\
&\quad \leqslant \left|G_{1-t^{-1}s}\left(x-\frac{y}{\sqrt{t}}\right)-G_1\left(x-\frac{y}{\sqrt{t}}\right)\right| + \left|G_1\left(x-\frac{y}{\sqrt{t}}\right)-G_1(x)\right|\\
&\quad \leqslant Ct^{-1}s\left|\partial_t G_{1-\theta t^{-1}s}\left(x-\frac{y}{\sqrt{t}}\right)\right| + Ct^{-1/2}\left|y\cdot\nabla G_1\left(x-\frac{\theta y}{\sqrt{t}}\right)\right|\\
&\quad \leqslant Ct^{-1}s\left|\partial_t G_{1-\theta t^{-1}s}\left(x-\frac{y}{\sqrt{t}}\right)\right| + C\delta\sup_x|\nabla G_1|. \qquad (6.14)
\end{aligned}
$$

Noticing that $||u(s)||_{L^{\alpha+1}}^{\alpha+1}$ is integrable over $[0, \infty)$ and that

$$
\begin{aligned}
&t^{-1} \int_0^{\delta t} s(1+s)^{-\alpha} ||\partial_t G_{1-\theta t^{-1}s}||_{L^p} \, ds \\
&\leqslant \frac{C}{1-\alpha} \delta(1+\delta t)^{-\alpha+1} + \frac{C}{(\alpha-1)(\alpha-2)} \left(t^{-1} - \frac{(1+\delta t)^{-\alpha+2}}{t} \right) \\
&\to 0, \quad \text{as } t \to \infty,
\end{aligned}
$$

we have for the first term of the right-hand side of (6.14) can be estimated as

$$
\begin{aligned}
I_1 &\leqslant \int_0^{\delta t} \int_{B_{\delta\sqrt{t}}} ||G_{1-t^{-1}s}(x - t^{-1/2}y) - G_1(x)||_{L^p} |u(s,y)|^{\alpha+1} \, dy \, ds \\
&\leqslant \delta \int_0^{\delta t} \int_{B_{\delta\sqrt{t}}} ||\nabla G_1||_{L^p} |u(s,y)|^{\alpha+1} \, dy \, ds \\
&\leqslant \delta \int_0^{\delta t} ||u(s)||_{L^{\alpha+1}}^{\alpha+1} \, ds \leqslant C\delta.
\end{aligned} \tag{6.15}
$$

For I_2, by

$$
\int_0^\infty \int_{\mathbb{R}^2} |u(s)|^{\alpha+1} \, dy \, ds < \infty,
$$

we see

$$
\int_0^\infty \int_{B_{\delta\sqrt{t}}} |u(s)|^\alpha u(s) \, dy \, ds \to 0
$$

for $t \to \infty$.

Finally for I_3,

$$
\begin{aligned}
I_3 \leqslant & \left|\left| \int_{\delta t}^t \int_{\mathbb{R}^2} G_{1-t^{-1}s}(x - t^{-1/2}y) |u(s,y)|^\alpha u(s,y) \, dy \, ds \right|\right|_{L^p} \\
& + \left|\left| \int_{\delta t}^t \int_{\mathbb{R}^2} G_1(x) |u(s,y)|^\alpha u(s,y) \, dy \, ds \right|\right|_{L^p} \\
\leqslant & \, C \int_{\delta t}^t ||G_{1-t^{-1}s}||_{L^p} ||u(s)||_{L^{\alpha+1}}^{\alpha+1} \, ds + ||G_1||_p \int_{\delta t}^t ||u(s)||_{L^{\alpha+1}}^{\alpha+1} \, ds.
\end{aligned} \tag{6.16}
$$

Here

$$\begin{aligned} \|G_{1-t^{-1}s}\|_{L^p}^p &= \int_{\mathbb{R}^2} \left(\frac{1}{1-t^{-1}s}\right)^p e^{-p|x-t^{-1/2}y|^2/(1-t^{-1}s)}\,dx \\ &= \left(\frac{1}{1-t^{-1}s}\right)^p \int_{\mathbb{R}^2} e^{-p|x-t^{-1/2}y|^2/(1-t^{-1}s)}\,dx \\ &\leqslant C\left(\frac{1}{1-t^{-1}s}\right)^{p-1}. \end{aligned} \tag{6.17}$$

Hence

$$\begin{aligned} I_3 &\leqslant C\int_{\delta t}^{t} \|G_{1-t^{-1}s}\|_{L^p}\|u(s)\|_{\alpha+1}^{\alpha+1}\,ds + C\int_{\delta t}^{t}(1+s)^{-\alpha}\,ds \\ &\leqslant Ct\int_{\delta}^{1}(1-s)^{-1+1/p}(1+ts)^{-\alpha}\,ds + C(1+\delta t)^{-\alpha+1} \\ &\leqslant C\left(\frac{t}{(1+\delta t)^{\alpha}}\right) + C(1+\delta t)^{-\alpha+1} \to 0 \quad \text{as } t\to\infty, \end{aligned} \tag{6.18}$$

where we used $p<\infty$ and $\alpha>2/n=1$. When $p=\infty$, we simply arrange the estimate in (6.18) by using the L^p-L^q estimate of the heat flow and yields the same conclusion. $\square$

Acknowledgments

The authors would like to express their sincere gratitude to Professor Kenji Nishihara for his many helpful suggestions. They are also grateful to Professor Nakao Hayashi and Professor Genivième Raugel for valuable discussion.

References

[1] H. Bellout, A. Friedman, Blow-up estimates for a nonlinear hyperbolic heat equation, Math. Z. 145 (1975) 251–254.
[2] J. Bergh, J. Löfström, Interpolation Spaces, Springer, Berlin, Heiderberg, New York, 1976.
[3] P. Brenner, On L^p-L^q estimate for the wave equation, SIAM J. Math. Anal. 20 (1989) 354–366.
[4] P. Brenner, V. Thomée, L.B. Wahlbin, Besov Spaces and Applications to Difference Methods for Initial Value Problems, Lecture Notes in Mathematics, Vol. 434, Springer, Berlin, Heidelberg, New York, 1975.
[5] A. Carpio, Large-time behavior in incompressible Navier–Stokes equations, SIAM J. Math. Anal. 27 (1996) 449–475.
[6] R. Courant, D. Hilbert, Methods of Mathematical Physics, Vol. II, Wiley, New York, 1962.
[7] M. Escobedo, E. Zuazua, Large time behavior for convection-diffusion equation in $\mathbb{R}^n$, J. Funct. Anal. 100 (1991) 119–161.
[8] Y. Fujigaki, T. Miyakawa, Asymptotic profiles of the nonstationary incompressible Navier–Stokes flows in the whole space, SIAM J. Math. Anal. 33 (3) (2001) 523–544.

[9] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t - \Delta u = u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sec. 1 13 (1966) 109–124.
[10] Th. Gallay, G. Raugel, Scaling variables and asymptotic expansions in damped wave equations, J. Differential Equations 150 (1998) 42–97.
[11] J. Ginibre, G. Velo, Generalized Strichartz inequalities for the wave equation in homogeneous Besov spaces, in: Nonlinear Wave 1995, GAKUTO International Series in Mathematical Science and Applications, Vol. 10, 1997, pp. 135–144.
[12] R. Ikehata, A remark on a critical exponent for the semilinear dissipative wave equation in the one dimensional half space, Differential Integral Equations 16 (2003) 727–736.
[13] R. Ikehata, Y. Miyaoka, T. Nakatake, Decay estimate of solutions for dissipative wave equation in $\mathbb{R}^N$ with lower power nonlinearities, J. Math. Soc. Japan 56 (2) (2004) 365–374.
[14] R. Ikehata, M. Ohta, Critical exponents for semilinear dissipative wave equations in $\mathbb{R}^N$, J. Math. Anal. Appl. 269 (1) (2002) 87–97.
[15] G. Karch, Selfsimilar profiles in large time asymptotics of solutions to damped wave equation, Studia Math. 143 (2000) 175–197.
[16] S. Kawashima, M. Nakao, K. Ono, On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term, J. Math. Soc. Japan 47 (1995) 617–653.
[17] T.-T. Li, Nonlinear heat conduction with finite speed of propagation, Proceedings of the China–Japan Symposium on Reaction–Diffusion Equations and their Applications an Computational Aspect, Shanghai, 1994, World Scientific Publishing, River Edge, NJ, 1997, pp. 81–91.
[18] T.-T. Li, Y. Zhou, Breakdown of solutions to $\Box u + u_t = u^{1+\alpha}$, Discrete Contin. Dynam. Systems 1 (1995) 503–520.
[19] P. Marcati, K. Nishihara, The $L^p - L^q$ estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media, J. Differential Equations 191 (2003) 445–469.
[20] B. Marshall, W. Strauss, S. Wainger, $L^p - L^q$ estimates for the Klein–Gordon equations, J. Math. Pures Appl. 59 (1980) 417–440.
[21] A. Matsumura, On the asymptotic behavior of solution of semi-linear wave equation, Publ. Res. Inst. Math. Sci. 12 (1976) 169–189; Funkcial. Ekvac. 26.
[22] M. Nakao, K. Ono, Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations, Math. Z. 214 (1993) 325–342.
[23] T. Narazaki, $L^p - L^q$ estimates for damped wave equation and their applications to semilinear problem, J. Math. Soc. Japan 56 (2) (2004) 585–626.
[24] K. Nishihara, Asymptotic behavior of solutions of quasilinear hyperbolic equations with linear damping, J. Differential Equations 137 (1997) 384–395.
[25] K. Nishihara, $L^p - L^q$ estimates of solutions to the damped wave equation in 3-dimensional space and their application, Math. Z. 244 (2003) 631–649.
[26] T. Ogawa, Well posedness and real analytic method for the nonlinear wave and dispersive equation, Rokko Lecture Note Series in Mathematics, Kobe University, to appear (in Japanese).
[27] K. Ono, Global existence and asymptotic behavior of small solutions for semilinear dissipative wave equations, Discrete Contin. Dynam. Systems 9 (2003) 651–662.
[28] I.E. Segal, Dispersion for non-linear relativistic equations, II, Ann. Sci. École Norm. Sup. (4) 1 (1968) 459–497.
[29] E.M. Stein, G. Weiss, Fourier Analysis in Euclidean Spaces, Princeton University Press, Princeton, NJ, 1971.
[30] W.A. Strauss, Nonlinear wave equations, CBMS Regional Conferences Series in Mathematics, American Mathematical Society Providence, RI, 1989.
[31] G. Todorova, B. Yordanov, Critical exponent for a nonlinear wave equation with damping, J. Differential Equations 174 (2000) 464–489.
[32] H. Yang, A. Milani, On the diffusion phenomenon of quasilinear hyperbolic waves, Bull Sci. Math. 124 (2000) 415–433.
[33] Q. Zhang, A blow-up result for a nonlinear wave equation with damping: the critical case, C. R. Acad. Sci. Paris 333 (2001) 109–114.